

DECAY ESTIMATES AND STRICHARTZ ESTIMATES OF FOURTH-ORDER SCHRÖDINGER OPERATOR

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ABSTRACT. We study time decay estimates of the fourth-order Schrödinger operator $H = (-\Delta)^2 + V(x)$ in \mathbb{R}^d for $d = 3$ and $d \geq 5$. We analyze the low energy and high energy behaviour of resolvent $R(H; z)$, and then derive the Jensen-Kato dispersion decay estimate and local decay estimate for $e^{-itH}P_{ac}$ under suitable spectrum assumptions of H . Based on Jensen-Kato decay estimate and local decay estimate, we obtain the $L^1 \rightarrow L^\infty$ estimate of $e^{-itH}P_{ac}$ in 3-dimension by Ginibre argument, and also establish the endpoint global Strichartz estimates of $e^{-itH}P_{ac}$ for $d \geq 5$. Furthermore, using the local decay estimate and the Georgescu-Larenas-Soffer conjugate operator method, we prove the Jensen-Kato type decay estimates for some functions of H .

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1. INTRODUCTION

In this paper we consider the time decay estimates of the operator

$$H = H_0 + V(x), \quad H_0 = (-\Delta)^2$$

in $L^2(\mathbb{R}^d)$ for $d = 3$ and $d \geq 5$, where $V(x)$ is a real valued bounded function as a multiplication operator. In the sequel, we assume that $V(x) = O(|x|^{-\beta})$ for large $|x|$ with some $\beta > 0$ (the specific β will be given in conclusions below).

It was known that the fourth-order Schrödinger equation was introduced by Karpman [32, 33] and Karpman and Shagalov [34] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. The nonlinear beam equation, or fourth-order wave equation has been involved in the study of plate and beams, see e.g. Love [40], in the study of interaction of water waves, see Bretherton [3], and in the study of the motion of a suspension bridge, see Lazer and MacKenna [35] and MacKenna and Walter [44, 45]. Recently, these fourth-order equations were considered in mathematics by many authors. For example, Levandosky and Strauss had considered the stability and instability of fourth-order solitary waves [37], the time decay estimates for fourth-order wave equations [38] and [39]. Moreover, the well-posedness and scattering problems of nonlinear fourth-order Schrödinger equation have been further studied by many authors now, see e.g. Miao, Xu and Zhao [46, 47], Pausader [48, 49], C. Hao, L. Hsiao and B. Wang [15, 16], Ruzhansky, B. Wang and H. Zhang [52], Segata [55, 56] and references therein.

In the studies of linear or nonlinear dispersive equations, one is faced with the need to quantitatively estimate the time decay of the solution in different kinds of norms. Indeed, many interesting estimates including *local decay estimates*, *Jensen-Kato decay estimates*, *L^p -decay estimates* and *Strichartz estimates*, play central roles in these studies. Note that all the papers we mentioned above are considering fourth-order linear or nonlinear equations. So the purpose of this paper is to establish such estimates for the fourth-order Schrödinger type operator $H = (-\Delta)^2 + V$ with some decay potential.

Our work is partially motivated by Jensen and Kato's famous work [20]. They proved the time decay estimates of $e^{-it(-\Delta+V)}P_{ac}$ in the weighted L^2 -norm. Furthermore, Murata [41] had generalized Jensen and Kato's work to the operator $P(D) + V$, where $P(D)$ is an m -order elliptic differential operator with real constant coefficients, assuming that the all critical points of polynomial $P(\xi)$ are non-degenerate, i.e.

$$(\nabla P)(\xi_0) = 0, \quad \det(\partial_i \partial_j P(\xi))|_{\xi_0} \neq 0.$$

However, the biharmonic operator $(-\Delta)^2$ does not satisfy this assumption at $\xi = 0$, thus Murata's method does not apply for $H = (-\Delta)^2 + V$. Hence in this paper, we first establish Jensen-Kato type decay estimate and local decay estimate for H , which are very important, for example to asymptotic completeness of the perturbed linear fourth-order Schrödinger equations. Secondly, based on the Jensen-Kato decay estimate and local decay estimate, we prove L^p -type decay estimates and endpoint Strichartz estimates for H , which can be then applied to the well-posedness problems, scattering theory and soliton asymptotic stability problems of the nonlinear fourth-order Schrödinger equation. Finally, we introduce the Georgescu-Larenas-Soffer conjugate operator method to derive the Jensen-Kato type estimate which starts only from the local decay estimate. Our methods differ from Murata, and apply to more general functions of the Laplacian, including $(-\Delta)^m (m \geq 2)$ with the degenerate original point.

We notice that, for the biharmonic operator $(-\Delta)^2$, Ben-Artzi, Koch and Saut [4] had proven the following sharp kernel estimate,

$$(1.1) \quad |D^\alpha I_0(t, x)| \leq C|t|^{-(d+|\alpha|)/4} \left(1 + |t|^{-1/4}|x|\right)^{(|\alpha|-d)/3}, \quad t \neq 0, x \in \mathbb{R}^d,$$

where $I_0(t, x)$ is the kernel of $e^{-it\Delta^2}$. The above estimate implies the $L^1 \rightarrow L^\infty$ -estimate of $e^{-it\Delta^2}$, namely

$$(1.2) \quad \|e^{-it\Delta^2}\|_{L^1(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)} \leq C|t|^{-d/4}.$$

Hence the endpoint Strichartz estimates for the free operator $(-\Delta)^2$ can be established, by using the $L^1 \rightarrow L^\infty$ -estimate (1.2) and Keel-Tao's arguments. Besides, the Jensen-Kato decay estimate, local decay estimate and other L^p -decay estimates of $e^{-it\Delta^2}$ can be directly derived from the decay estimate (1.2).

For the fourth-order Schrödinger operator $H = (-\Delta)^2 + V$, it is much more difficult to establish the similar kernel estimate (1.1) for e^{-itH} , and difficult to prove the $L^1 \rightarrow L^\infty$ -estimate (1.2). In order to prove L^p -decay estimates and Strichartz estimates of e^{-itH} , we start from Jensen-Kato decay estimate and local decay estimate of H to overcome difficulties caused by the potential V . A key point to get the Jensen-Kato decay estimate, is the asymptotic behaviour of the spectral density $E'(\lambda)$ of H near zero. In the first part of the paper, we deduce asymptotic expansion in the weighted Sobolev spaces $\mathcal{H}_\sigma^s(\mathbb{R}^d)$ for resolvent $R(H; z) = (H - z)^{-1}$ and $E'(\lambda)$ around zero for dimensions $d = 3$ and $d \geq 5$ assuming that *zero is a regular point for H* (see Definition 2.5 below). Our strategy is using the following free resolvent identity to get the asymptotic resolvent expansion:

$$(1.3) \quad R(H_0; z) = (H_0 - z)^{-1} = \frac{1}{2z^{1/2}} [(-\Delta - z^{1/2})^{-1} - (-\Delta + z^{1/2})^{-1}], \quad z \in \mathbb{C} \setminus [0, +\infty).$$

Here and in other places, we denote the resolvent of T by $R(T; z) = (T - z)^{-1}$.

Since the leading term of the resolvent expansion depends on the dimension d , we deal with three cases separately: $d = 3$, $d \geq 5$ and odd, $d \geq 6$ and even. The following are three typical expansions of $R(H; z)$ for $d = 3, 5, 6$ as $|z| \rightarrow 0$ (with appropriate choice of weighted function w):

$$(1.4) \quad d = 3, \quad w(H - z)^{-1}w = C_0 + z^{1/4}C_1 + z^{2/4}C_2 + \dots$$

$$(1.5) \quad d = 5, \quad R(H, z) = B_0 + \frac{1-i}{2}z^{1/4}B_1 + \frac{1+i}{2}z^{3/4}B_2 + (-1)zB_3 + \dots$$

$$(1.6) \quad d = 6, \quad R(H, z) = B_1^0 + z^{1/2}B_2^1 + z \ln z^{1/2}B_3^1 + zB_2^{1,1} + \dots$$

where $z^{1/4}$ is in the first quadrant of complex plane. The expansions are valid in the operator norm in $\mathcal{B}(\mathcal{H}_\sigma^{-2}(\mathbb{R}^d), \mathcal{H}_\sigma^2(\mathbb{R}^d))$, where $\mathcal{H}_\sigma^s(\mathbb{R}^d)$ is the weighted Sobolev space with the associated norm

$$\|u\|_{\mathcal{H}_\sigma^s(\mathbb{R}^d)} = \| \langle x \rangle^\sigma \langle i\nabla \rangle^s u \|_{L^2(\mathbb{R}^d)}.$$

And $L_\sigma^2(\mathbb{R}^d)$ denotes the space $\mathcal{H}_\sigma^s(\mathbb{R}^d)$ when $s = 0$, i.e. $L_\sigma^2(\mathbb{R}^d) = \mathcal{H}_\sigma^0(\mathbb{R}^d)$. Here $s, \sigma, \sigma' \in \mathbb{R}$ and $\langle x \rangle = (1 + |x|^2)^{1/2}$. In general, the expansions to higher orders require larger β and σ, σ' .

In order to establish the Jensen-Kato decay estimate for e^{-itH} , we also need to study the high energy decay properties of $R(H; z)$, see Subsection 2.3. In fact, the high energy estimate is easier than the low energy estimate. For Schrödinger operator, for instance, in Kopylova and Komech [29] one can find the high energy decay of the free and perturbed resolvent in the weighted

Sobolev norms in 3-dimension. For the constant coefficients differential operator $P(D)$ of order m and of principal type, Agmon first established such estimate in the fundamental work [1]. Moreover, Murata had also established this kind of estimate for first order pseudo-differential operators [42] and higher order elliptic operators [43]. For the fourth-order Schrödinger operator H , our method is using the results of free resolvent $R(-\Delta; \zeta)$ and the resolvent identity (1.3) to get the high energy decay estimates of $R((-\Delta)^2; z)$ directly, and then to get high energy derivative estimate of $R^{(k)}(H; z)$ for any $k \geq 0$. Our decay rate of $R^{(k)}(H; z)$ is $-(3 + 3k)/4$, which is compatible with Agmon's result if $k = 0$.

In this paper, we always assume that *there are no positive embedded eigenvalues and 0 is a regular point of H* . For the Schrödinger operators, the fact about the absence of positive eigenvalue was first shown in Kato's work [31] if the potential is continuous and decay $O(|x|^{-\beta})$ at infinity for some $\beta > 1$. Since then, the classical result has been extended to Schrödinger operators with rough integrable potentials by several authors (see e. g. Jerison and Kenig [19], Kenig, Ruiz and Sogge [26], Ionescu and Jerison [18], Koch and Tataru [28] and references therein). Their basic strategy is proving the new Carleman estimate and unique continuation theorem, and then showing the absence of positive eigenvalues. The difficulty to follow these ideas for fourth-order Schrödinger operator is to establish suitable Carleman type estimate for $(-\Delta)^2$ and suitable form of unique continuation theorem for H . At present, a general criterion about absence of positive eigenvalues in higher order cases is not yet available except that V is a small potential, see e.g. [57]. Furthermore, we remark that Froese and Herbst's approach [9, 10] is more general than the works mentioned above, where they use the Mourre estimate of the Schrödinger operator and the positive preserving property of $e^{-it(-\Delta+V)}$. However, for $H = (-\Delta)^2 + V$, the positive preserving property of semigroup is an clear obstacle, even in the free case $e^{-it\Delta^2}$, see e. g. Reed and Simon [51, Theorem XIII. 53].

Now we state one of our main results: *Jensen-Kato decay estimate* (see Section 3 for the local decay estimate of $e^{-it(\Delta^2+V)}$). In the following discussion, all the constants C are allowed to depend on the dimension d , and to vary from line to line.

Theorem 1.1. *Let $H = (-\Delta)^2 + V$ with $\langle x \rangle^\beta V(x) \in L^\infty(\mathbb{R}^d)$ for some $\beta > 0$, and assume that H has no positive embedded eigenvalues and 0 is a regular point for H . Then the following conclusions hold:*

(i) *If $d = 3$ and $\beta > 11 + 3/2$, then for any $\sigma > 0$ we have*

$$(1.7) \quad \|e^{-itH} P_{ac} u\|_{L^2_{-\sigma}(\mathbb{R}^3)} \leq C \langle t \rangle^{-5/4} \|u\|_{L^2_{\sigma}(\mathbb{R}^3)}, \quad t \in \mathbb{R};$$

(ii) *If $d \geq 5$, d odd and $\beta > d$, then for any $\sigma > d/2$ we have*

$$(1.8) \quad \|e^{-itH} P_{ac} u\|_{L^2_{-\sigma}(\mathbb{R}^d)} \leq C \langle t \rangle^{-d/4} \|u\|_{L^2_{\sigma}(\mathbb{R}^d)}, \quad t \in \mathbb{R};$$

(iii) *If $d \geq 6$, d even and $\beta > d + 4$, then the above estimate (1.8) holds again for any $\sigma > d/2 + 2$.*

Here $L^2_{\sigma}(\mathbb{R}^d)$ is the weighted Sobolev space, and P_{ac} denotes the projection onto the absolutely continuous spectrum space of H . The constants C depend on the dimension d only.

In the second part of this paper, we apply the local decay estimate and Jensen-Kato decay estimate to derive the L^p -type estimate (Ginibre argument) and Strichartz estimates for H . The unpublished argument of Ginibre for Schrödinger operator in three or higher dimensions allows

passing from the local decay to global decay, in the form of a $L^1 \cap L^2 \rightarrow L^2 + L^\infty$. For Schrödinger operator, such result is

$$\|e^{-it(-\Delta+V)}P_{ac}u\|_{L^2+L^\infty(\mathbb{R}^d)} \leq C(d)\langle t \rangle^{-d/2}\|u\|_{L^1 \cap L^2(\mathbb{R}^d)}.$$

Here P_{ac} is the projection onto the continuous spectrum space of $-\Delta + V$, see e.g. W. Schlag [54] and the references therein. We remark that, the first stronger L^p -time decay estimate than Ginibre argument above for Schrödinger operator is due to Journé, Soffer and Sogge's work [25], where they established the $L^1 \rightarrow L^\infty$ -decay estimate using some important cancellation lemmas (see e.g. Lemma 2.2 of [25]).

For the biharmonic operator $(-\Delta)^2$, we have known that the $L^1 \rightarrow L^\infty$ -decay estimate (1.2) holds from Ben-Artzi, Koch and Saut's work [4]. For the perturbed fourth-order Schrödinger operator $H = (-\Delta)^2 + V$, there are few results about the $L^1 \rightarrow L^\infty$ time decay estimate of $e^{-itH}P_{ac}$ in any dimension up to now. The method in Journé, Soffer and Sogge [25] can not be simply applied to $H = (-\Delta)^2 + V$, as some similar cancellation lemmas involving $e^{-it\Delta^2}Ve^{it\Delta^2}$ are much more complicated than Laplacian $-\Delta$. In this paper, based on the free decay estimate (1.2), we can use Ginibre argument for $e^{-itH}P_{ac}$ with decay $\langle t \rangle^{-d/4}$ for $d \geq 5$, and also obtain the $L^1 \rightarrow L^\infty$ time decay estimate of $e^{-itH}P_{ac}$ with decay $|t|^{-1/2}$ for $d = 3$, though it is not optimal.

Theorem 1.2. *Let $d = 3$ and H satisfy the same conditions as given in Theorem 1.1. Then we have*

$$(1.9) \quad \|e^{-itH}P_{ac}\|_{L^1(\mathbb{R}^3) \rightarrow L^\infty(\mathbb{R}^3)} \leq \begin{cases} C|t|^{-3/4}, & \text{for } 0 < |t| < 1, \\ C|t|^{-1/2}, & \text{for } |t| \geq 1, \end{cases}$$

where P_{ac} denotes the projection onto the absolutely continuous spectrum space of H .

Theorem 1.3. *Let $d \geq 5$ and H satisfy the same conditions as given in Theorem 1.1. Then we have*

$$(1.10) \quad \|e^{-itH}P_{ac}u\|_{L^2+L^\infty(\mathbb{R}^d)} \leq C\langle t \rangle^{-d/4}\|u\|_{L^2 \cap L^1(\mathbb{R}^d)}, \quad t \in \mathbb{R},$$

where P_{ac} denotes the projection onto the absolutely continuous spectrum space of H .

The studies of space-time integrability properties of the solutions for Schrödinger equations and the corresponding inhomogeneous equation, have been pursued by many authors in the last thirty years. The matter of fact is given by the Strichartz estimates, which have become fundamental and amazing tools for the studies of PDEs including the well-posedness and scattering theory, see e.g. [6, 27, 58, 59, 60]. For the nonlinear fourth-order Schrödinger equation without potential, see [46, 47] for some global well-posedness and scattering results in both focusing and defocusing cases. Here we will establish Strichartz estimates for the following fourth-order Schrödinger equation with potential V and source term $h(t)$:

$$(1.11) \quad \begin{cases} i\partial_t \Psi = (\Delta^2 + V)\Psi + h(t), \\ \Psi(0, \cdot) = \Psi_0 \in L^2(\mathbb{R}^d). \end{cases}$$

In the case of $d = 3$, we get local Strichartz type estimate by Keel-Tao's method [27]. For $d \geq 5$, based on Jensen-Kato decay estimate (1.8) and local decay estimate (3.1), we prove the following global endpoint Strichartz estimate. Recall that *the admissible pair (q, r) for the fourth-order Schrödinger equation satisfies*

$$(1.12) \quad \frac{4}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leq q \leq \infty, \quad d \geq 5.$$

Theorem 1.4. *Consider the equation (1.11). Let H satisfy the same conditions as given in Theorem 1.1 for $d \geq 5$. Then for any admissible pairs (q, r) and (\tilde{q}, \tilde{r}) , we have the homogeneous Strichartz estimate*

$$(1.13) \quad \|e^{-itH} P_{ac} \Psi_0\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \leq C(d) \|\Psi_0\|_{L^2(\mathbb{R}^d)},$$

and the dual homogeneous Strichartz estimate

$$(1.14) \quad \left\| \int_{\mathbb{R}} e^{isH} P_{ac} h(s, \cdot) ds \right\|_{L_x^q(\mathbb{R}^d)} \leq C(d) \|h\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R} \times \mathbb{R}^d)}.$$

Furthermore, the solution $\Psi(t, x)$ satisfies that

$$(1.15) \quad \|P_{ac} \Psi(t, x)\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \leq C(d) \|\Psi_0\|_{L^2} + \|h\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R} \times \mathbb{R}^d)},$$

where P_{ac} is the projection onto the absolutely continuous spectrum of H .

At the end of this paper, we use the Georgescu-Larenas-Soffer conjugate operator method to get Jensen-Kato type decay estimate for e^{-itH} and $e^{-it\sqrt{H+m^2}}$. For the free half-wave operator $e^{-it\sqrt{H_0+1}}$, W. Chen, C. Miao and X. Yao [7] had proven the $L^p \rightarrow L^q$ -estimates using the kernel of $e^{-it\sqrt{H_0+1}}$. The conjugate operator method here we used reveals that the local decay estimate implies the Jensen-Kato decay estimate. The idea of this method is to construct the Larenas-Soffer conjugate operator $\tilde{A} = A + B$, where $A = -\frac{i}{2}(x \cdot \nabla + \nabla \cdot x)$ and B is a bounded operator. The commutator of H and A equals $q(H) + K$, where $q(H)$ is a function of H and K is a good operator in some sense. The new conjugate operator \tilde{A} keeps the same good properties of A , and kills the tail K when \tilde{A} does commute with H . The difficulty of using this approach [12] is to prove the C^k -condition, i.e. $H \in C^k(A)$ and $\sqrt{H+m^2} \in C^k(A)$. We remark that this method relies on the local decay estimate to show the existence of operator B and then allows to get higher decay estimate in an easier way than the Jensen-Kato expansion method, see [2, 11, 12, 36].

Notations. In what follows, we write $A \lesssim B$ to signify that there exists a constant C such that $A \leq CB$. And *odd* $d \geq 5$ means $d \geq 5$ and d is odd. similarly, for *even* $d \geq 6$.

2. THE RESOLVENT $R(H; z)$ OF $H = (-\Delta)^2 + V$

By the spectral theorem, we know

$$e^{-itH} P_{ac} = \int_0^\infty e^{-it\lambda} E'(\lambda) d\lambda.$$

In order to get Jensen-Kato decay estimate, we need to analyze the property of $E'(\lambda)$ and higher order derivatives $E^{(k)}(\lambda)$ for λ small and large. This is related to the asymptotic properties of the resolvent $R(H; z)$ for z small and large since $\pi E'(\lambda) = \text{Im } R(H; \lambda + i0)$.

In this section, we aim to obtain the low energy asymptotic expansion and the high energy decay estimate of $R(H; z)$. For z large, we prove the high energy decay of $R(H; z)$ in $\mathcal{B}(L_\sigma^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$ directly. For z near zero, we use the Born splitting

$$(2.1) \quad R(H; z) = (1 + R(H_0; z)V)^{-1} R(H_0; z)$$

to derive the asymptotic expansion in the weighted Sobolev space. At the end, we show that for $\lambda > 0$, the limit $\lim_{\epsilon \downarrow 0} R(H; \lambda \pm i0)$ exists in $\mathcal{B}(L_\sigma^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$ to obtain the asymptotic property of $E'(\lambda)$ for λ small and decay of $E'(\lambda)$ for λ large.

Recall that, for the Schrödinger operator $-\Delta + V$, Kato and Jensen derived resolvent expansion of $(-\Delta + V - \zeta)^{-1}$ around zero in dimensions $d \geq 3$ by using the Born decomposition

$$(2.2) \quad R(-\Delta + V; \zeta) = (1 + R(-\Delta; \zeta)V)^{-1}R(-\Delta; \zeta).$$

For the free resolvent $R(-\Delta; \zeta)$, they use the kernel $k(x, y; \zeta)$ of $(-\Delta - \zeta)^{-1}$ to derive the asymptotic expansion, where

$$(2.3) \quad k(x, y; \zeta) = \frac{i}{4} \left(\frac{\zeta^{1/2}}{2\pi|x-y|} \right)^{d/2-1} H_{d/2-1}^{(1)}(\zeta^{1/2}|x-y|), \quad \text{Im } \zeta^{1/2} \geq 0,$$

and $H_{d/2-1}^{(1)}$ is the first Hankel function. Thus, the key point is to get the expansion of $(1 + R(-\Delta; \zeta)V)^{-1}$ near zero, see [20, 21, 22]. For the cases $d \geq 3$, since the free resolvent $R(-\Delta; \zeta)$ has no singularity at zero, the expansion of resolvent $(-\Delta + V - \zeta)^{-1}$ has no terms with negative power of z nor $\ln z$ separately. While for $d = 1$ there exists $\zeta^{-1/2}G_{-1}$ and for $d = 2$ there exists $\ln \zeta G_{0,-1}$, the classical Born splitting in [22] couldn't be useful in dimensions $d = 1, 2$. So Jensen and Nenciu [23] developed a unified approach to deal with these cases.

For the fourth-order Schrödinger operator $H = (-\Delta)^2 + V$, we know that the singularity of $R((-\Delta)^2; z)$ at $z = 0$ in d -dimensions ($d \geq 3$) is the same as $R(-\Delta; \zeta)$ in $(d-2)$ -dimensions. Therefore, in the case $d = 3$, we can apply the unified approach of Jensen and Nenciu [23] similarly done for Schrödinger operators of 1 and 2-dimensions. For $d \geq 5$, we will follow the original one [22].

2.1. Asymptotic expansion of free resolvent $R(H_0; z)$ near $z = 0$. For the asymptotic expansion of free resolvent, the direct way is expanding its kernel. Actually, one can calculate the kernel of $R(H_0; z)$ by dividing the integrand into two parts and using the Cauchy integral directly. For instance, the explicit kernel $K(x, y)$ of $R(H_0; z)$ in $d = 3$ is as follows:

$$K(x, y; z) = \frac{1}{8\pi} \frac{e^{iz^{1/4}|x-y|} - e^{-z^{1/4}|x-y|}}{z^{1/2}|x-y|}, \quad \text{Re } z^{1/4} > 0, \quad \text{Im } z^{1/4} > 0.$$

For other dimensions, one can also using the same dividing trick to give the kernel of $R(H_0; z)$ by Hankel function with $z \in \mathbb{C} \setminus [0, +\infty)$. Here, we make use of the asymptotic expansion of free resolvent $R(-\Delta; \zeta)$ of Schrödinger operator, and apply the resolvent splitting (1.3) to get the asymptotic expansion of $R(H_0; z)$ with z around zero.

Recall the asymptotic expansion of $R(-\Delta; \zeta)$ around zero, see e. g. [20, 21, 22], we have:

Lemma 2.1. *For $\zeta \in \mathbb{C} \setminus [0, +\infty)$ and $\text{Im } \zeta^{1/2} > 0$, we have the following formal expansion of the resolvent of free Laplacian $R(-\Delta; \zeta)$ as $\zeta \rightarrow 0$:*

$$(2.4) \quad R_0(-\Delta; \zeta) = \sum_{j=0}^{\infty} (i\zeta^{1/2})^j G_j^{\text{odd}}, \quad \text{odd } d \geq 3;$$

$$(2.5) \quad R_0(-\Delta; \zeta) = \sum_{j=0}^{\infty} \sum_{k=0}^1 \zeta^j (\ln \zeta)^k G_j^{k, \text{even}}, \quad \text{even } d \geq 6.$$

For $j = 0, 1, 2, \dots$, G_j^{odd} are operators given by the following integral kernels

$$(2.6) \quad G_j^{\text{odd}}(x, y) = \frac{(-1)^{(d-3)/2}}{2(2\pi)^{(d-1)/2}} d_j |x-y|^{j-(d-2)},$$

with $d_j = \sum_{k=0, k \geq (d-3)/2-j}^{(d-3)/2} \frac{((d-3)/2+k)!}{k!((d-3)/2-k)!} \frac{(-2)^{-k}}{(k+j-(d-3)/2)!}$. Especially, $d_j = 1$ when $d = 3$.

For $j = 0, 1, 2, \dots, d/2 - 2$, $G_j^{k, \text{even}}$ are operators given by the following integral kernels

$$(2.7) \quad G_j^{0, \text{even}}(x, y) = \pi^{-d/2} \frac{(d/2 - j - 2)!}{j!} 4^{-j-1} |x - y|^{2j+2-d},$$

$$(2.8) \quad G_j^{1, \text{even}}(x, y) = 0.$$

For $j \geq d/2 - 1$, $G_j^{k, \text{even}}$ are operators given by the following integral kernels

$$(2.9) \quad \begin{aligned} G_j^{0, \text{even}}(x, y) &= (4\pi)^{-d/2} [\vartheta(j+1) + \vartheta(j+2-d/2)] \frac{(-1/4)^{j+1-d/2}}{j!(d/2-1+j)!} |x-y|^{2j+2-d} \\ &\quad - 2(4\pi)^{-d/2} (-1/4)^{j+1-d/2} \frac{\ln(|x-y|/2)}{j!(d/2-1+j)!} |x-y|^{2j+2-d} \\ &\quad + \frac{i}{4} (4\pi)^{-d/2+1} \frac{1}{j!(d/2-1+j)!} (-1/4)^{j+1-d/2} |x-y|^{2j+2-d}. \end{aligned}$$

$$(2.10) \quad G_j^{1, \text{even}}(x, y) = -(4\pi)^{-m/2} \frac{(-1/4)^{j+1-d/2}}{j!(d/2-1+j)!} |x-y|^{2j+2-d}.$$

Here $\vartheta(k)$ is given by $\vartheta(1) = -1$, $\vartheta(k) = \sum_{j=1}^{k-1} 1/j - \mathbb{E}$, and \mathbb{E} is the Euler's constant.

Remark 2.2. $d_j = 0$ for j odd and $0 < j < d-2$, see [21, Lemma 3.3]. Further, all the $G_j^{\text{odd}}, G_j^{k, \text{even}} \in \mathcal{B}(\mathcal{H}_\sigma^{-2}, \mathcal{H}_{\sigma'}^2)$ with σ, σ' depend on j . The proof details and more properties of G_j^{odd} and $G_j^{k, \text{even}}$, please see [20, 21]. One difference is that [20, 21] proved $G_j^{\text{odd}}, G_j^{k, \text{even}} \in \mathcal{B}(\mathcal{H}_\sigma^0, \mathcal{H}_{\sigma'}^0)$ and then using the identity

$$(1 - \Delta)R(-\Delta; \zeta) = 1 + (1 + \zeta)R(-\Delta; \zeta)$$

to improve into $\mathcal{B}(\mathcal{H}_\sigma^{-1}, \mathcal{H}_{\sigma'}^1)$. Here for fourth-order Schrödinger operator, we can improve into $\mathcal{B}(\mathcal{H}_\sigma^{-2}, \mathcal{H}_{\sigma'}^2)$ by the same way, since we have

$$(1 + \Delta^2)R(H_0; z) = 1 + (1 + z)R(H_0; z).$$

Based on the expansion of $R(-\Delta; \zeta)$ and the resolvent identity (1.3), we obtain the formal expansions of $R(H_0; z)$ directly. For simplifying the notation, we let $z = \mu^4$ and choose μ in the first quadrant of the complex plane i.e. $\text{Re } \mu > 0$ and $\text{Im } \mu > 0$. Note that if μ in the first quadrant, then $z \in \mathbb{C} \setminus [0, +\infty)$.

Lemma 2.3. For μ in the first quadrant of the complex plane, we have the formal expansions of the resolvent of free fourth-order Schrödinger operator $R(H_0; z)$:

$$(2.11) \quad R(H_0; \mu^4) = \sum_{j=0}^{\infty} \frac{i^j - (-1)^j}{2} \mu^{j-2} G_j^{\text{odd}}, \text{ odd } d \geq 3;$$

$$(2.12) \quad R(H_0; \mu^4) = \sum_{j=0}^{\infty} \sum_{k=0}^1 \frac{1}{2} (\ln \mu^2)^k \mu^{2j-2} \tilde{G}_j^{k, \text{even}}, \text{ even } d \geq 6.$$

Here for all $j \in \mathbb{N}$, $\tilde{G}_j^{0, \text{even}} = \frac{1-(-1)^j}{2} G_j^{0, \text{even}} + \frac{(-1)^j i \pi}{2} G_j^{1, \text{even}}$ and $\tilde{G}_j^{1, \text{even}} = \frac{1-(-1)^j}{2} G_j^{1, \text{even}}$.

Now we give a strict meaning for the above formal expansions. The formal series (2.11) and (2.12) are an asymptotic expansions for $\mu \rightarrow 0$ in the following sense.

Proposition 2.4. *For $z \in \mathbb{C} \setminus [0, +\infty)$, we have in $\mathcal{B}(\mathcal{H}_\sigma^{-2}(\mathbb{R}^d), \mathcal{H}_{\sigma'}^2(\mathbb{R}^d))$ the following asymptotic expansions as $z \rightarrow 0$.*

(i) For $d \geq 3$ and d odd,

$$(2.13) \quad R(H_0; z) = \sum_{j=0}^N \frac{i^j - (-1)^j}{2} z^{(j-2)/4} G_j^{odd} + o(z^{(N-2)/4}),$$

with σ and σ' satisfy:

- 1) for $0 \leq N \leq (d-3)/2$: $\sigma, \sigma' > 1/2$ and $\sigma + \sigma' > (d+1)/2$;
- 2) for $(d-3)/2 < N \leq d-3$: $\sigma, \sigma' > N+2-d/2$ and $\sigma + \sigma' > N+2$;
- 3) for $N \geq d-2$: $\sigma, \sigma' > N+2-d/2$.

(ii) For $d \geq 6$ and d even,

$$(2.14) \quad R(H_0; z) = \sum_{j=0}^N \sum_{k=0}^1 \frac{1}{2} (\ln z^{1/2})^k z^{(j-1)/2} \tilde{G}_j^{k, even} + o(z^{\frac{N-1}{2}} \ln z^{1/2}),$$

with σ and σ' satisfy:

- 1) for $0 \leq N \leq (d-3)/4$: $\sigma, \sigma' > 1/2$ and $\sigma + \sigma' > (d+1)/2$;
- 2) for $(d-3)/4 < N < d/2 - 1$: $\sigma, \sigma' > 2N+2-d/2$ and $\sigma + \sigma' > 2N+2$;
- 3) for $N \geq d/2 - 1$: $\sigma, \sigma' > 2N+2-d/2$.

Furthermore, the expansion can be differentiated in z any number of times. More precisely, the r -th derivative of the finite series in (2.13) (Res. (2.14)) up to $j = N$, is equal to $(d/dz)^r R(H_0; z)$ up to an error $o(z^{(N-2)/4-r})$ (Res. $o(z^{\frac{N-1}{2}-r} \ln z^{1/2})$) in the norm of $\mathcal{B}(\mathcal{H}_\sigma^{-2}(\mathbb{R}^d), \mathcal{H}_{\sigma'}^2(\mathbb{R}^d))$ with σ, σ' satisfy the relationships with N as given in the above Lemma 2.4.

Proof. The proof details please see [20, Lemma 2.3] and [21, Lemma 3.5, Lemma 3.9], since we derive the expansion around zero of $R(H_0; z)$ by the expansion of $R(-\Delta; \zeta)$. \square

Notice that there are factors as $i^j - (-1)^j$ and $1 - (-1)^j$ in the expansions of $R(H_0; z)$, so many terms can be cancelled. For $d = 3$, the first two terms are $\frac{1}{4\pi}\mu^{-1} + (-G_2^{d=3})$ and $G_2^{d=3}(x, y) = 1/(4\pi)|x-y|^{4-3}$, so $z = 0$ is a singularity point of $R(H_0; z)$. For $d \geq 5$, the lowest power of μ is positive since there are many terms equal zero and the zero term depends on the dimension d . For odd $d \geq 5$, the first term is

$$-G_2^{odd}(x, y) = \frac{(-1)^{(d-1)/2}}{2(2\pi)^{(d-1)/2}} d_2 |x-y|^{4-d},$$

and for even $d \geq 6$, the first term is

$$G_1^{0, even}(x, y) = \frac{(d/2-3)!}{16\pi^{d/2}} |x-y|^{4-d}.$$

They are both the convolution kernel of the Riesz potential $(-\Delta)^{-2}$ in \mathbb{R}^d respectively, see Stein [53]. Further, [21, Lemma 2.3] implies that $(-\Delta)^{-2} \in \mathcal{B}(\mathcal{H}_\sigma^{-2}, \mathcal{H}_{\sigma'}^2)$ with $d \geq 9$ and $\sigma + \sigma' \geq 4$.

2.2. Asymptotic expansion of $R(H; z)$ near $z = 0$. We deal with the expansion of $R(H; z)$ in $d = 3$ and $d \geq 5$ separately, since zero is a singular point of $R(H_0; z)$ in 3-dimensions while $d \geq 5$ not. For these two cases, we both use the Born decomposition but different form. Before getting the expansion, we should analyze the zero point of H .

In order to avoid stating some results separately for d even and d odd, we use the following notation:

$$G_0(x, y) = \begin{cases} -G_2^{odd}(x, y) = \frac{(-1)^{(d-1)/2}}{2(2\pi)^{(d-1)/2}} d_2 |x - y|^{4-d}, & \text{odd } d \geq 3; \\ G_1^{0,even}(x, y) = \frac{(d/2-3)!}{16\pi^{d/2}} |x - y|^{4-d}, & \text{even } d \geq 6. \end{cases}$$

Let $\beta > 4$, then $H = H_0 + V$ is a selfadjoint operator defined as the quadratic form sum of H_0 and V . For $0 \leq s \leq \beta$, we define

$$\mathcal{N} = \left\{ f \in L_{-s}^2(\mathbb{R}^d) \mid (1 + G_0 V)f = 0 \right\}, \quad \mathcal{M} = \left\{ f \in L_s^2(\mathbb{R}^d) \mid (1 + V G_0)f = 0 \right\}.$$

A priori \mathcal{N} and \mathcal{M} may depend on s , but \mathcal{N} and \mathcal{M} are obviously monotone in s in opposite directions, and also dual to each other, so $\dim \mathcal{N} = \dim \mathcal{M} < \infty$ shows that \mathcal{N} and \mathcal{M} are independent of s for $0 \leq s \leq \beta$.

Definition 2.5. We say that u is a zero resonant state if $u \in \mathcal{N} \setminus L^2(\mathbb{R}^d)$. We say that zero is a regular point of H if $\mathcal{N} = \{0\}$.

Remark 2.6. Following the same argument as in part of Jensen [21], one can prove that there is no zero energy resonance of H for $d \geq 9$. Actually, in dimensions $d \geq 9$, [21, Lemma 2.4] shows that G_0 is nice in the sense that G_0 is bounded from $L^2(\mathbb{R}^d)$ to $L_4^2(\mathbb{R}^d)$. By the identity $R(H; z)(1 + VR(H_0; z)) = R(H_0; z)$, taking $z = 0$ as with $d \geq 9$, then $R(H; 0)(1 + VG_0) = G_0 = (-\Delta)^{-2}$. As a consequence there is no zero resonance for H in dimensions $d \geq 9$ since G_0 is defined on L^2 .

For the 3-dimensional case, we start from the symmetrized resolvent formula

$$(2.15) \quad (H - \mu^4)^{-1} = (H_0 - \mu^4)^{-1} - (H_0 - \mu^4)^{-1} v M(\mu)^{-1} v (H_0 - \mu^4)^{-1},$$

where

$$(2.16) \quad v(x) = |V(x)|^{\frac{1}{2}}, \quad U(x) = \begin{cases} 1, & V(x) \geq 0, \\ -1, & V(x) < 0. \end{cases}, \quad M(\mu) = U + v(H_0 - \mu^4)^{-1} v.$$

Let $w(x) = U(x)v(x)$, then from the identity

$$(2.17) \quad (1 - w(H - \mu^4)^{-1} v) (1 + w(H_0 - \mu^4)^{-1} v) = 1$$

we have

$$(2.18) \quad w(H - \mu^4)^{-1} w = U - M(\mu)^{-1}.$$

Notice that, if we take $|v|$ as the weight function instead of the classic one $\langle x \rangle^\sigma$, then we clearly, define the unusual weighted spaces $L_v^2(\mathbb{R}^3)$ by

$$\|\psi\|_{L_v^2(\mathbb{R}^3)} := \|v(x)\psi(x)\|_{L^2(\mathbb{R}^3)} < \infty.$$

In order to get the expansion of $(H - \mu^4)^{-1}$ in the unusual weighted space $L_v^2(\mathbb{R}^3)$, it suffices to obtain the expansion of $M(\mu)^{-1}$ in $L^2(\mathbb{R}^3)$. Note that $M(\mu)$ has known expansion in powers of μ up to an order depending upon decay rate of V at infinity, hence the problem is to prove that $M(\mu)^{-1}$ also has expansion in powers of μ up to some order and to compute the coefficients. Applying the unified approach, under our spectral assumptions we have:

Theorem 2.7. *Let $\langle x \rangle^\kappa V(x) \in L^2(\mathbb{R}^3)$ for κ large enough and p be the largest integer satisfying $\kappa > 2p + 5$. Assume that H has no positive embedded eigenvalues and 0 is a regular point for H . Then for μ in the first quadrant of the complex plane, there exists $\mu_0 > 0$ such that for $|\mu| \leq \mu_0$, $w(H - \mu^4)^{-1}w$ has the expansion in $\mathcal{B}(L^2(\mathbb{R}^3), L^2(\mathbb{R}^3))$,*

$$(2.19) \quad wR(H; \mu^4)w = U - Qm_0^{-1}Q + \sum_{j=1}^{p-1} M'_j \mu^j + \mu^p \mathfrak{R}(\mu).$$

Here $\mathfrak{R}(\mu)$ is uniformly bounded and the coefficients M'_j can be computed explicitly.

For the proof, we refer the readers to see Appendix A1 of this paper. Here we need to point out that we first get the expansion of $U - M(\mu)^{-1}$ in $\mathcal{B}(L^2(\mathbb{R}^3), L^2(\mathbb{R}^3))$, and then by identity (2.18) get the asymptotic property of $(H - \mu^4)^{-1}$ in the usual weighted Sobolev space with $v\langle x \rangle^\sigma \in L^\infty(\mathbb{R}^3)$ using the following trick :

$$w\langle x \rangle^\sigma \langle x \rangle^{-\sigma} (H - \mu^4)^{-1} \langle x \rangle^{-\sigma} \langle x \rangle^\sigma w = U - M(\mu)^{-1}.$$

Moreover, if potential V satisfies $\langle x \rangle^\beta V(x) \in L^\infty(\mathbb{R}^3)$ with $\beta > \kappa + 3/2$, which implies the condition that $\langle x \rangle^\kappa V(x) \in L^2(\mathbb{R}^3)$.

For the cases of $d \geq 5$, we use the Born decomposition (2.1) to get the expansion of $R(H; z)$. It's enough to get the expansion of $(1 + R(H_0; z)V)^{-1}$. Proposition 2.4 and Remark 2.2 imply that

$$1 + R(H_0; z)V = 1 - G_2^{odd}V + \sum_{j=3}^N \frac{i^j - (-1)^j}{2} z^{(j-2)/4} G_j^{odd}V + o(z^{(N-2)/4}), \quad \text{odd } d \geq 5;$$

$$1 + R(H_0; z)V = 1 + G_1^{0,even}V + \sum_{j=2}^N \sum_{k=0}^1 \frac{1}{2} (\ln z^{1/2})^k z^{(j-1)/2} \tilde{G}_j^{k,even}V + o(z^{(N-1)/2} \ln z^{1/2}), \quad \text{even } d \geq 6.$$

The assumption that zero is neither an eigenvalue of H nor a resonance of H imply that the function $(1 + R(H_0; 0)V)u = 0$ only admits zero solution, here $R(H_0; 0) = (-\Delta)^{-2}$. Thus we can expand $(1 + R(H_0; z)V)^{-1}$ (Neumann series) for small z and then get the expansion of $R(H; z)$ near zero.

Proposition 2.8. *Assume that 0 is a regular point for H . We have in $\mathcal{B}(\mathcal{H}_{-\sigma}^2(\mathbb{R}^d), \mathcal{H}_{-\sigma}^2(\mathbb{R}^d))$ the expansion as $z \in \mathbb{C} \setminus \mathbb{R}$ and $z \rightarrow 0$.*

(i) For $d \geq 5$ and d odd,

$$(2.20) \quad (1 + R(H_0; z)V)^{-1} = (1 - G_2^{odd}V)^{-1} + \sum_{j=3}^N \frac{(-1)^j - i^j}{2} z^{\frac{j-2}{4}} C_j + o(z^{\frac{N-2}{4}})$$

with β and σ are assumed to satisfy:

- 1) for $3 \leq N \leq (d-3)/2$: $\beta > (d+1)/2$ and $1/2 < \sigma < \beta - 1/2$;
- 2) for $(d-3)/2 < N \leq d-3$: $\beta > N+2$, $N+2-d/2 < \sigma < \beta - (N+2-d/2)$;
- 3) for $N \geq d-2$: $\beta > 2N+4-d$ and $N+2-d/2 < \sigma < \beta - (N+2-d/2)$.

All the C_j can be calculated through the Newmann series.

(ii) For $d \geq 6$ and d even,

$$(2.21) \quad (1 + R(H_0; z)V)^{-1} = (1 + G_1^{0, \text{even}}V)^{-1} + \sum_{j=2}^N \sum_{k=0}^{\varsigma(j)} z^{\frac{j-1}{2}} (\ln z^{1/2})^k C_j^k + o(z^{\frac{L-1}{2}} (\ln z^{1/2})^{\varsigma(N)})$$

with β and σ are assumed to satisfy:

1) for $2 \leq N \leq (d-3)/4$: $\beta > (d+1)/2$ and $1/2 < \sigma < \beta - 1/2$;

2) for $(d-3)/4 < N < d/2 - 1$: $\beta > 2N + 2$, $2N + 2 - d/2 < \sigma < \beta - (2N + 2 - d/2)$;

3) for $N \geq d/2 - 1$: $\beta > 4N + 4 - d$ and $2N + 2 - d/2 < \sigma < \beta - (2N + 2 - d/2)$.

Here $\varsigma(j) \in \{0, 1\}$ and C_j^k can be computed by the Neumann series.

Theorem 2.9. Under the same assumptions of Proposition 2.8, for $z \in \mathbb{C} \setminus \mathbb{R}$ and $z \rightarrow 0$ we have:

(i) For $d \geq 5$ and d odd,

$$(2.22) \quad R(H, z) = B_0 + \sum_{j=3}^N \frac{i^j - (-1)^j}{2} z^{(j-2)/4} B_j + o(|z|^{(N-2)/4}).$$

Here $B_0 = (1 - G_2^{\text{odd}}V)^{-1}(-G_2^{\text{odd}})$ and $B_j = 0$ for $3 \leq j < d-2$ and d odd.

(ii) For $d \geq 6$ and d even,

$$(2.23) \quad R(H, z) = B_1^0 + \sum_{j=1}^N \sum_{k=0}^{\rho(j)} (z^{1/2})^{j-1} (\ln z^{1/2})^k B_j^k + o(|z|^{(N-1)/2} (\ln z^{1/2})^{\rho(N)}),$$

and $B_1^0 = (1 + G_1^{0, \text{even}}V)^{-1}G_1^{0, \text{even}}$.

All the other B_j , B_j^k and $\rho(N) \in \{0, 1\}$ can be calculated explicit by the product of expansion series of $R(H_0; z)$ and $(1 + R(H_0; z)V)^{-1}$.

Remark 2.10. For the 4-dimensions case, Jensen and Nenciu's unified approach can be applied to get the expansion of resolvent of the fourth-order Schrödinger operator near zero but it's very complicated even for Schrödinger operator. Notice that we deduce the expansion under the assumption that 0 is a regular point of H . If zero is a simple eigenvalue or resonance of H , one can also get the expansion by the Born decomposition by the same argument as one for Schrödinger operator [20, 21]. However, it would be hard and interesting to further analyze the full structure of the zero eigenspace of the fourth-order Schrödinger operator.

Proposition 2.11. The asymptotic expansions for $R(H; z)$ above can be differentiated in z any times, in the sense that

$$(2.24) \quad \left(\frac{d}{dz}\right)^r [wR(H; z)w - (U - Qm_0^{-1}Q + \sum_{j=1}^{p-1} M'_j \mu^j)] = o(|z|^{p/4-r}), \quad d = 3;$$

$$(2.25) \quad \left(\frac{d}{dz}\right)^r [R(H; z) - (B_0 + \sum_{j=3}^N \frac{i^j - (-1)^j}{2} z^{(j-2)/4} B_j)] = o(|z|^{(N-2)/4-r}), \quad \text{odd } d \geq 5;$$

$$(2.26) \quad \left(\frac{d}{dz}\right)^r \left[R(H; z) - (B_1^0 + \sum_{j=1}^N \sum_{k=0}^{\rho(j)} z^{\frac{j-1}{2}} (\ln z^{1/2})^k B_j^k) \right] = o(|z|^{\frac{N-1}{2}-r} (\ln z^{1/2})^{\rho(N)}), \quad \text{even } d \geq 6.$$

Proof. To see this, for $d \geq 5$ note that $R(H; z) = (1 + R(H_0; z)V)^{-1}R(H_0; z)$, in which $R(H_0; z)$ has differentiable asymptotic series (Proposition 2.4). Since the product of two differentiable asymptotic series is differentiable, it suffices to show that the asymptotic series for $(1 + R(H_0; z)V)^{-1}$ is differentiable. This is seen from

$$(d/dz)(1 + R(H_0; z)V)^{-1} = -(1 + R(H_0; z)V)^{-1}R'(H_0; z)V(1 + R(H_0; z)V)^{-1},$$

due to the result just mentioned about the product of two asymptotic series. For $d = 3$, note that $w(H - \mu^4)^{-1}w = U - M(\mu)^{-1}$ and

$$(d/dz)[wR(H; z)w] = M^{-1}(\mu)vR'(H_0; z)vM^{-1}(\mu).$$

Higher order derivatives can be done similarly. Note that we do the derivative in the topology of $\mathcal{B}(L_\sigma^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$. In order to ensure that $(d/dz)^r R(H; z) \in \mathcal{B}(L_\sigma^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$, σ should be larger than the case $r = 0$. Precisely, we need $\sigma > r + 1/2$ which will be shown in the Theorem 2.20 below. \square

2.3. High energy decay estimates of $R(H; z)$. The following result is the fundamental Agmon-Kato estimate on decay of the Schrödinger resolvent operator for complex z goes to the infinity in the weighted Sobolev norms. It plays a crucial role in time-decay estimates of the solution to Schrödinger equation.

Lemma 2.12. ([29, Theorem 16.1]) *For $\zeta \in \mathbb{C} \setminus [0, +\infty)$, $k = 0, 1, 2, 3, \dots$, any $\sigma > k + \frac{1}{2}$ and any $a > 0$, the bound*

$$(2.27) \quad \|R^{(k)}(-\Delta; \zeta)\|_{\mathcal{H}_\sigma^s(\mathbb{R}^3) \rightarrow \mathcal{H}_{-\sigma}^s(\mathbb{R}^3)} \leq C(\sigma, a)|\zeta|^{-(1+k)/2}, \quad |\zeta| \geq a,$$

holds for $s \in \mathbb{R}$.

We note that the proof of Theorem 16.1 in [29] does not depend on the dimension d , so that for free Schrödinger operator in d -dimensions, the estimate (2.27) also holds. The following is the similar conclusion for $H_0 = (-\Delta)^2$.

Proposition 2.13. *For $z \in \mathbb{C} \setminus [0, +\infty)$, $k = 0, 1, 2, 3, \dots$, any $\sigma > k + \frac{1}{2}$ and any $a > 0$, the bound*

$$(2.28) \quad \|R^{(k)}(H_0; z)\|_{\mathcal{H}_\sigma^s(\mathbb{R}^d) \rightarrow \mathcal{H}_{-\sigma}^s(\mathbb{R}^d)} \leq C(\sigma, a)|z|^{-(3+3k)/4}, \quad |z| \geq a$$

holds for $s \in \mathbb{R}$.

Proof. First we prove decay (2.28) for $k = 0$. We aim to prove that for $\sigma > \frac{1}{2}$,

$$(2.29) \quad \|R(H_0; z)\|_{\mathcal{H}_\sigma^s(\mathbb{R}^d) \rightarrow \mathcal{H}_{-\sigma}^s(\mathbb{R}^d)} \leq C(\sigma, a)|z|^{-3/4}, \quad |z| > a, \quad z \in \mathbb{C} \setminus [0, +\infty).$$

By Lemma 2.12, we have

$$\begin{aligned} \|R(H_0; z)\|_{\mathcal{H}_\sigma^s \rightarrow \mathcal{H}_{-\sigma}^s} &= \frac{1}{2|\mu^2|} \|R(-\Delta; \mu^2) - R(-\Delta; (i\mu)^2)\|_{\mathcal{H}_\sigma^s \rightarrow \mathcal{H}_{-\sigma}^s} \\ &\leq \frac{1}{2|\mu^2|} \left(\|R(-\Delta; \mu^2)\|_{\mathcal{H}_\sigma^s \rightarrow \mathcal{H}_{-\sigma}^s} + \|R(-\Delta; (i\mu)^2)\|_{\mathcal{H}_\sigma^s \rightarrow \mathcal{H}_{-\sigma}^s} \right) \\ &\leq C(\sigma, a)|z|^{-3/4}. \end{aligned}$$

Now we check decay estimate (2.28) for $k \neq 0$. For $R(H_0; z)$ we have the recurrent relations

$$(2.30) \quad zR^{(k)}(H_0; z) = -kR^{(k-1)}(H_0; z) + \frac{1}{4}[x \cdot \nabla, R^{(k-1)}(H_0; z)].$$

By an induction process we get the estimate (2.28). \square

Now, we prove the high energy decay estimate of perturbed resolvent $R^{(k)}(H; z)$. The proof relies on estimate (2.28) for the free resolvent $R(H_0; z)$ and on some useful identities for $R(H_0; z)$ and $R(H; z)$. Denote V_0 to be the minimal value of $V(x)$, i.e. $V_0 = \min\{V(x), x \in \mathbb{R}^d\}$. Since for any $\lambda \in \mathbb{R} \setminus [V_0, +\infty)$, $H - \lambda = \Delta^2 + V - \lambda > 0$, then for the resolvent set $\rho(H)$ of H , we have $\mathbb{C} \setminus [V_0, +\infty) \subseteq \rho(H)$.

Lemma 2.14. *Let $V(x)$ satisfies that $\langle x \rangle^\beta V(x) \in L^\infty(\mathbb{R}^d)$ with some $\beta > 0$. Then operators $R(H_0; z)V$, $VR(H_0; z) \in L^2(\mathbb{R}^d)$ are compact for $z \in \mathbb{C} \setminus [0, +\infty)$. And $1 + R(H_0; z)V$, $1 + VR(H_0; z)$ are invertible in $L^2(\mathbb{R}^d)$ for $z \in \mathbb{C} \setminus [V_0, +\infty)$.*

Proof. The proof relies on the Hermitian symmetry and the Fredholm theorem. We divide the proof into two parts. First, we show that $R(H_0; z)V$ and $VR(H_0; z)$ are compact operators. Then we use the Fredholm theorem to get the invertibility of $1 + R(H_0; z)V$. The invertibility of $1 + VR(H_0; z)$ follows by the duality. The proof details please see the Appendix A2. \square

Proposition 2.15. *Let $k = 0, 1, 2, 3, \dots$, and assume that the potential $V(x)$ satisfies $\langle x \rangle^\beta V(x) \in L^\infty(\mathbb{R}^d)$ with $\beta > k + 1$. Then for large $z \in \mathbb{C} \setminus [V_0, +\infty)$ and any $\sigma > k + 1/2$, the bound*

$$(2.31) \quad \|R^{(k)}(H; z)\|_{\mathcal{H}_\sigma^s(\mathbb{R}^d) \rightarrow \mathcal{H}_{-\sigma}^s(\mathbb{R}^d)} \leq C(\sigma, k)(|z|^{-(3+3k)/4})$$

holds for any $s \in \mathbb{R}$.

Proof. By the Born decomposition formula we have

$$(2.32) \quad R(H; z) = [1 + R(H_0; z)V]^{-1}R(H_0; z), \quad R(H; z) = [1 + VR(H_0; z)]^{-1}R(H_0; z).$$

The identities (2.32) imply

$$(2.33) \quad [1 + R(H_0; z)V]R(H; z) = R(H_0; z), \quad R(H; z)[1 + VR(H_0; z)] = R(H_0; z).$$

Differentiating (2.33) k times, we obtain

$$(2.34) \quad \begin{aligned} R^{(k)}(H; z) &= R(H_0; z)^{(k)} - \sum_{k_1+k_2=k-1} \frac{(k-1)!}{k_1!k_2!} R^{(k_1)}(H; z) VR^{(k_2+1)}(H_0; z) \\ &\quad - \sum_{k_1+k_2=k-1} \frac{(k-1)!}{k_1!k_2!} R^{(k_1+1)}(H_0; z) VR^{(k_2)}(H; z) \\ &\quad + \sum_{k_1+k_2+k_3=k-1} \frac{(k-1)!}{k_1!k_2!k_3!} R^{(k_1)}(H; z) VR^{(k_2+1)}(H_0; z) VR^{(k_3)}(H; z). \end{aligned}$$

Note that, if $\sigma_1 < \sigma_2$, then estimate (2.31) holds for σ_1 implies (2.31) holds for σ_2 . This follows by $\|f_1\|_{L_{\sigma_1}^2} \leq \|f_1\|_{L_{\sigma_2}^2}$ and $\|f_2\|_{L_{-\sigma_2}^2} \leq \|f_2\|_{L_{-\sigma_1}^2}$.

For $k = 0$, it is enough to prove estimate (2.31) for $\sigma \in (\frac{1}{2}, \frac{\beta}{2}]$. By the Born splitting, (2.31) holds if the norm of the inverse operator $[1 + R(H_0; z)V]^{-1} : \mathcal{H}_{-\sigma}^s \rightarrow \mathcal{H}_{-\sigma}^s$ is uniform bounded in z for large $z \in \mathbb{C} \setminus [0, +\infty)$, since $\|R(H_0; z)\|_{\mathcal{H}_\sigma^s \rightarrow \mathcal{H}_{-\sigma}^s} \leq C(|z|^{-3/4})$ by decay estimate (2.28) with

$k = 0$. Now, we aim to show $[1 + R(H_0; z)V]^{-1} : \mathcal{H}_{-\sigma}^s \rightarrow \mathcal{H}_{-\sigma}^s$ is uniform bounded in z for large $z \in \mathbb{C} \setminus [0, +\infty)$. It is equivalent to prove for large $z \in \mathbb{C} \setminus [0, +\infty)$,

$$(2.35) \quad \|g\|_{\mathcal{H}_{-\sigma}^s} \leq C\|(1 + R(H_0; z)V)g\|_{\mathcal{H}_{-\sigma}^s}.$$

In fact, by the triangle inequality we have

$$(2.36) \quad \left| \|g\|_{\mathcal{H}_{-\sigma}^s} - \|R(H_0; z)Vg\|_{\mathcal{H}_{-\sigma}^s} \right| \leq \|(1 + R(H_0; z)V)g\|_{\mathcal{H}_{-\sigma}^s} \leq \|g\|_{\mathcal{H}_{-\sigma}^s} + \|R(H_0; z)Vg\|_{\mathcal{H}_{-\sigma}^s}.$$

Then for $\|R(H_0; z)Vg\|_{\mathcal{H}_{-\sigma}^s}$, by the decay estimate (2.28) we have

$$\|R(H_0; z)Vg\|_{\mathcal{H}_{-\sigma}^s} \leq C|z|^{-3/4}\|Vg\|_{\mathcal{H}_{\sigma}^s} \leq C|z|^{-3/4}\|\langle x \rangle^{2\sigma}V\|_{L^\infty}\|g\|_{\mathcal{H}_{-\sigma}^s},$$

and $\langle x \rangle^{2\sigma}V \in L^\infty$ since $\sigma \in (1/2, \beta/2]$. Thus for z large enough, we have

$$\|R(H_0; z)Vg\|_{\mathcal{H}_{-\sigma}^s} \leq \frac{1}{4}\|g\|_{\mathcal{H}_{-\sigma}^s},$$

hence (2.35) holds by (2.36). Notice that the constant C in (2.35) do not depend on z .

We prove the estimates (2.31) for $k \geq 1$ by induction. Namely, assume (2.31) holds for $R^{(j)}(H; z)$ with $j = 0, 1, 2, \dots, k-1$. Consider the second summand on the right hand side of (2.34). Choosing $\sigma' \in (k_1 + \frac{1}{2}, \beta - \frac{3}{2} - k_2)$ (it is possible since $\beta > k+1$), we obtain

$$\begin{aligned} & \|R^{(k_1)}(H; z)VR^{(k_2+1)}(H_0; z)\psi\|_{\mathcal{H}_{-\sigma}^s} \\ & \leq C(|z|^{-(3+3k_1)/4})\|VR^{(k_2+1)}(H_0; z)\psi\|_{\mathcal{H}_{\sigma'}^s} \\ & \leq C(|z|^{-(3+3k_1)/4})\|R^{(k_2+1)}(H_0; z)\psi\|_{\mathcal{H}_{\sigma'-\beta}^s} \\ & \leq C(|z|^{-(6+3k)/4})\|\psi\|_{\mathcal{H}_{\sigma'}^s} \end{aligned}$$

since $\beta - \sigma' > k_2 + 1 + \frac{1}{2}$.

The third summand can be estimated similarly by choosing $\sigma' \in (k_1 + \frac{3}{2}, \beta - \frac{3}{2} - k_2)$.

Finally, consider the last summand. Taking $\sigma' \in (k_1 + k_3 + \frac{1}{2}, \beta - \frac{3}{2} - k_2)$, we get

$$\begin{aligned} & \|R^{(k_1)}(H; z)VR^{(k_2+1)}(H_0; z)VR^{(k_3)}(H; z)\psi\|_{\mathcal{H}_{-\sigma}^s} \\ & \leq C(|z|^{-(3+3k_1)/4})\|VR^{(k_2+1)}(H_0; z)VR^{(k_3)}(H; z)\psi\|_{\mathcal{H}_{\sigma'}^s} \\ & \leq C(|z|^{-(3+3k_1)/4})\|R^{(k_2+1)}(H_0; z)VR^{(k_3)}(H; z)\psi\|_{\mathcal{H}_{\sigma'-\beta}^s} \\ & \leq C(|z|^{-(6+3k_1+3k_2)/4})\|VR^{(k_3)}(H; z)\psi\|_{\mathcal{H}_{\beta-\sigma'}^s} \\ & \leq C(|z|^{-(6+3k_1+3k_2)/4})\|R^{(k_3)}(H; z)\psi\|_{\mathcal{H}_{-\sigma'}^s} \\ & \leq C(|z|^{-(6+3k)/4})\|\psi\|_{\mathcal{H}_{\sigma'}^s} \end{aligned}$$

since $\sigma' > k_1 + \frac{1}{2}$, $\beta - \sigma' > k_2 + 1 + \frac{1}{2}$ and $\sigma' > k_3 + \frac{1}{2}$. □

2.4. Limiting absorption principle of $R(H; z)$. The limiting absorption principle was known in the diffraction theory for wave and Maxwell equations. It means the existence and continuity of the resolvent in the continuous spectrum. The continuity of the resolvent for Schrödinger operator in the weighted Sobolev norms was established by Agmon [1]. Hörmander also considered such problem for the general real coefficient self-adjoint operator $P(D)$. See [14, Chapter XIV]. Here we prove the continuity of $R(H, z)$ up to the positive real line in order to get the behaviour of $E'(\lambda)$ for $\lambda > 0$.

Denote by $\mathbb{C}^+ = \{\operatorname{Im} z > 0\}$ the open upper half complex plane, and by $\mathbb{C}^- = \{\operatorname{Im} z < 0\}$ the open lower half complex plane. Define Ξ be the disjoint union of \mathbb{C}^+ and \mathbb{C}^- with the identified points $z \leq 0$. For the resolvent of free Schrödinger operator $R(-\Delta; \zeta)$, summarizing the limiting absorption principle results of [20, 21, 22], or see Ginibre and Moulin [13] and Kuroda [30], we have:

Lemma 2.16. ([20, Theorem 8.1]) *Let $k = 0, 1, 2, \dots$. If $\sigma > k + 1/2$, then $R^{(k)}(-\Delta; \zeta) \in \mathcal{B}(L_\sigma^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$ is continuous in $\zeta \in \Xi \setminus \{0\}$. Further, the boundary value*

$$R^{(k)}(-\Delta; \lambda \pm i0) = \lim_{\epsilon \downarrow 0} R^{(k)}(-\Delta; \lambda \pm i\epsilon) \in \mathcal{B}(L_\sigma^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$$

exists for any $\lambda \in (0, +\infty)$. The decay estimate (2.27) can be extended from $\zeta \in \mathbb{C} \setminus [0, +\infty)$ to $\zeta \in \Xi \setminus \{0\}$.

Notice that the function $1/\mu^2$ is analytic for $\mu \in \mathbb{C} \setminus \{0\}$, and the fact that the difference of two analytic functions is also analytic. Here we need to point out that $z \in \mathbb{C} \setminus [0, +\infty)$ implies that $\mu^2, (i\mu)^2 \in \mathbb{C} \setminus [0, +\infty)$. By the resolvent identity (1.3), for the resolvent of free fourth order Schrödinger operator $R(H_0; z)$, we have:

Corollary 2.17. *Let $k = 0, 1, 2, \dots$. For $\sigma > k + 1/2$, then $R^{(k)}(H_0; z) \in \mathcal{B}(L_\sigma^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$ is continuous in $z \in \Xi \setminus \{0\}$. Further, the boundary value*

$$R^{(k)}(H_0; \lambda \pm i0) = \lim_{\epsilon \downarrow 0} R^{(k)}(H_0; \lambda \pm i\epsilon) \in \mathcal{B}(L_\sigma^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$$

exists for any $\lambda \in (0, +\infty)$, and the bound

$$(2.37) \quad \|R^{(k)}(H_0; z)\|_{L_\sigma^2(\mathbb{R}^d) \rightarrow L_{-\sigma}^2(\mathbb{R}^d)} = O(|z|^{-(3+3k)/4})$$

holds as $z \rightarrow \infty$ in $\Xi \setminus \{0\}$.

Next, under the spectral assumption of H that there is no positive embedded eigenvalues, we will prove that the boundary value $R(H; \lambda \pm i0)$ exists on $\lambda \in (0, +\infty)$. Recall that V_0 is the minimal value of potential function V . If $V_0 \geq 0$, then the segment $[V_0, 0] = \{0\}$.

Lemma 2.18. *Let $V(x)$ satisfies that $\langle x \rangle^\beta V(x) \in L^\infty(\mathbb{R}^d)$ with some $\beta > 1$. Then for $\sigma > 1/2$ and $\lambda > 0$, $R(H_0; \lambda \pm i0)V : L_{-\sigma}^2(\mathbb{R}^d) \rightarrow L_{-\sigma}^2(\mathbb{R}^d)$ and $VR(H_0; \lambda \pm i0) : L_\sigma^2(\mathbb{R}^d) \rightarrow L_\sigma^2(\mathbb{R}^d)$ are compact.*

Proof. Indeed, the compactness of $R(H_0; \lambda \pm i0)V$ follows by the Sobolev compact embedding $\mathcal{H}_{-\sigma'}^4(\mathbb{R}^d) \rightarrow L_{-\sigma}^2(\mathbb{R}^d)$ with $\sigma' \in (1/2, \min\{\sigma, \beta - \sigma\})$, see [29, Theorem 2.5]. Multiplication operator $V : L_{-\sigma}^2(\mathbb{R}^d) \rightarrow L_{-\sigma'}^2(\mathbb{R}^d)$ is continuous since $\sigma + \sigma' < \beta$. For the case $k = 0$ in Corollary 2.17, using the identity $(1 + \Delta^2)R(H_0; z) = 1 + (1 + z)R(H_0; z)$, we can improve $L_{-\sigma}^2(\mathbb{R}^d)$ to $\mathcal{H}_{-\sigma'}^4(\mathbb{R}^d)$. Thus $R(H_0; \lambda \pm i0) : L_\sigma^2(\mathbb{R}^d) \rightarrow \mathcal{H}_{-\sigma'}^4(\mathbb{R}^d)$ is also continuous by Corollary 2.17. The compactness of $VR(H_0; \lambda \pm i0)$ follows by the duality argument. \square

Theorem 2.19. *Let $V(x)$ satisfies that $\langle x \rangle^\beta V(x) \in L^\infty(\mathbb{R}^d)$ with some $\beta > 1$. Assume that H has no positive embedded eigenvalues. Then for $\sigma > 1/2$, $R(H; z) \in \mathcal{B}(L_\sigma^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$ is continuous in $z \in \Xi \setminus [V_0, 0]$. Further, the boundary value*

$$R(H; \lambda \pm i0) = \lim_{\epsilon \downarrow 0} R(H; \lambda \pm i\epsilon) \in \mathcal{B}(L_\sigma^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$$

exists for $\lambda \in (0, +\infty)$.

Proof. The analytic of $R(H; z)$ follows by (2.34). By Corollary 2.17 and Born splitting if

$$(1 + R(H_0; \lambda \pm i\epsilon)V)^{-1} \rightarrow (1 + R(H_0; \lambda \pm i0)V)^{-1}, \quad \epsilon \downarrow 0,$$

in the norm of $\mathcal{B}(L_{\sigma}^2, L_{-\sigma}^2)$. The convergence holds if and only if both limit operators $1 + R(H_0; \lambda \pm i0)V : L_{-\sigma}^2 \rightarrow L_{-\sigma}^2$ are invertible for $\lambda > 0$. According to the Fredholm Theorem and Proposition 2.18, it suffices to prove that for $\psi \in L_{-\sigma}^2$ with $\sigma \in (1/2, \beta - 1/2)$ the equations

$$(2.38) \quad [1 + R(H_0; \lambda \pm i0)V]\psi = 0$$

admit only zero solution. Notice that (2.38) implies that $\psi = R(H_0; \lambda \pm i0)f$ with $f = -V\psi$. The assumption on V with $\beta > 1$ implies that $f \in L_{\sigma'}^2$, where $\sigma' = \beta - 1/2$. So that ψ is λ -incoming or outgoing. Then by [14, Theorem 14.5.2], for any $s \in \mathbb{R}$ we have

$$\int \langle x \rangle^s |\psi|^2 dx < \infty,$$

which implies that $\psi \in L^2$. While ψ satisfies that

$$(2.39) \quad (H - \lambda)\psi = (H_0 - \lambda)(1 + R(H_0; \lambda \pm i0)V)\psi = 0.$$

Hence ψ is an eigenfunction related to the positive eigenvalue λ which is a contradiction to the assumption that the absence of positive embedded eigenvalues of H , thus $\psi = 0$. \square

Theorem 2.20. *Let $k = 0, 1, 2, 3, \dots$, and assume that $V(x)$ satisfies $\langle x \rangle^\beta V(x) \in L^\infty(\mathbb{R}^d)$ with $\beta > k + 1$. Under the assumption that H has no positive embedded eigenvalues and for any $\sigma > k + 1/2$, then $R^{(k)}(H; z) \in \mathcal{B}(L_{\sigma}^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$ is continuous for $z \in \Xi \setminus [V_0, 0]$. Further, the decay estimate (2.31) can be extended from $z \in \Xi \setminus [V_0, +\infty)$ to $z \in \Xi \setminus [V_0, 0]$, i.e. the bound*

$$(2.40) \quad \|R^{(k)}(H; z)\|_{L_{\sigma}^2(\mathbb{R}^d) \rightarrow L_{-\sigma}^2(\mathbb{R}^d)} = O(|z|^{-(3+3k)/4})$$

holds as $z \rightarrow \infty$ in $\Xi \setminus [V_0, 0]$.

Proof. Note that Theorem 2.19 implies the case $k = 0$ holds. Next we aim to show the case $k = 1$. Other cases hold by the induction process.

By the iterated identity (2.34), let $k = 1$ we have

$$(2.41) \quad \begin{aligned} R'(H; z) &= R'(H_0; z) - R(H; z)VR'(H_0; z) \\ &\quad - R'(H_0; z)VR(H; z) + R(H; z)VR'(H_0; z)VR(H; z). \end{aligned}$$

Thus by Theorem 2.17 and Theorem 2.19, for $\lambda > 0$, we know the limit

$$R'(H; \lambda \pm i0) = \lim_{\epsilon \downarrow 0} R'(H; \lambda \pm i\epsilon)$$

exists in $\mathcal{B}(L_{\sigma}^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$ with $\sigma > 3/2$.

Now, for the case $k = 1$, it is remaining to show

$$\|R'(H; \lambda \pm i0)\|_{L_{\sigma}^2(\mathbb{R}^d) \rightarrow L_{-\sigma}^2(\mathbb{R}^d)} = O(|\lambda|^{-(3+3)/4}), \quad \text{as } \lambda \rightarrow +\infty.$$

Indeed, by the Born splitting of $R(H; z)$, we have

$$(2.42) \quad \begin{aligned} R'(H; z) &= [1 + R(H_0; z)V]^{-1}R'(H_0; z) \\ &\quad - [1 + R(H_0; z)V]^{-1}R'(H_0; z)[1 + R(H_0; z)V]^{-1}R(H_0; z). \end{aligned}$$

Thus by Theorem 2.17 and the proof of Theorem 2.19, we know that

$$(2.43) \quad \begin{aligned} R'(H; \lambda \pm i0) &= [1 + R(H_0; \lambda \pm i0)V]^{-1} R'(H_0; \lambda \pm i0) \\ &\quad - [1 + R(H_0; \lambda \pm i0)V]^{-1} R'(H_0; z) [1 + R(H_0; \lambda \pm i0)V]^{-1} R(H_0; \lambda \pm i0). \end{aligned}$$

Following the same argument as in Proposition 2.15, then $[1 + R(H_0; \lambda \pm i0)V]^{-1}$ is uniform bounded in the norm of $\mathcal{B}(L^2_{-\sigma}(\mathbb{R}^d), L^2_{-\sigma}(\mathbb{R}^d))$ with $\sigma > 1/2$ for $\lambda > 0$ large enough. To summarize, we have shown that estimate (2.40) holds in the case $k = 1$. \square

Behaviours of the spectral density $E'(\lambda)$. Since $\pi E'(\lambda) = \text{Im } R(H; \lambda + i0)$, by Theorem 2.20 it follows that for any $k = 0, 1, 2, \dots$, $E^{(k+1)}(\lambda) \in \mathcal{B}(L^2_{\sigma}(\mathbb{R}^d), L^2_{-\sigma}(\mathbb{R}^d))$ is continuous in $\lambda \in (0, +\infty)$ for suitable large σ . Furthermore, in order to obtain the Jensen-Kato decay estimate (see Theorem 1.1), it is necessary to know the endpoint behaviour of $E'(\lambda)$ and high order derivatives $E^{(k+1)}(\lambda)$. Indeed, for small λ , the asymptotic expansion for $E'(\lambda) = \frac{1}{\pi} \text{Im} R(\lambda + i0)$ can be deduced immediately from those asymptotic expansion for $R(H; z)$ near zero in the preceding section. For large λ , we make use of the high energy decay estimate and continuity of $R^{(k)}(H; z)$.

Proposition 2.21. *Let $H = (-\Delta)^2 + V$ with $\langle x \rangle^{\beta} V(x) \in L^{\infty}(\mathbb{R}^d)$ for some $\beta > 0$, and assume that H has no positive embedded eigenvalues and 0 is a regular point for H . Then the following conclusions hold in $\mathcal{B}(L^2_{\sigma}(\mathbb{R}^d), L^2_{-\sigma}(\mathbb{R}^d))$ as $\lambda \rightarrow 0$:*

(i) *If $d = 3$ and $\beta > 11 + 3/2$, then for any $\sigma > 0$ we have*

$$(2.44) \quad E'(\lambda) = \lambda^{1/4} + \lambda^{1/2} + o(\lambda^{1/2}),$$

and the expansion can be differentiated 2 times.

(ii) *If odd $d \geq 5$ and $\beta > d$, then for any $\sigma > d/2$ we have*

$$(2.45) \quad E'(\lambda) = \lambda^{d/4-1} + o(\lambda^{d/4-1}),$$

and the expansion can be differentiated $d - 3$ times.

(iii) *If even $d \geq 6$ and $\beta > d + 4$, then for any $\sigma > d/2 + 2$ we have*

$$(2.46) \quad E'(\lambda) = \lambda^{d/4-1} + o(\lambda^{d/4} \ln \lambda^{\frac{1}{2}}),$$

and the expansion can be differentiated $d/2 - 1$ times.

Here, the above asymptotic expansion of $E'(\lambda)$ can be differentiated in λ in the same sense as in Proposition 2.11.

Proof. Since $\pi E'(\lambda) = \text{Im } R(H; \lambda + i0)$, then the continuity of $R^{(k)}(H; z)$ implies that $E'(\lambda)$ is differentiable. Recall that there are many terms cancelled in the expansion of $R(H_0; z)$. In order to get an error term, we need to choose $p = 3$ in expansion (2.19), thus the result (2.44) holds. Similarly, (2.45) follows from Theorem 2.9 with choosing $N = d - 2$ and (2.46) follows from Theorem 2.9 with choosing $N = d/2$ to get an error term. Note that for the even case, we choose $N = d/2$ not $d/2 - 1$, the purpose is that to get the error $o(\lambda^{d/4} \ln \lambda^{\frac{1}{2}})$. The chosen relationship between β and σ can be seen from the expansion results of Proposition 2.8 and Theorem 2.7.

The differentiability is established by the same sense as what we did for the differentiability of $R(H; z)$ in Proposition 2.11. \square

Proposition 2.22. *For $\lambda \rightarrow +\infty$, let $k = 0, 1, 2, 3, \dots$. Assume that $V(x)$ satisfies $\langle x \rangle^\beta V(x) \in L^\infty(\mathbb{R}^d)$ with $\beta > k + 1$. Under the assumption that H has no positive embedded eigenvalues and for any $\sigma > k + 1/2$, we have in the norm of $\mathcal{B}(L_\sigma^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$,*

$$(2.47) \quad E^{(k+1)}(\lambda) = O(\lambda^{-3(k+1)/4}), \quad \lambda \rightarrow +\infty.$$

Proof. For large $\lambda > 0$, this proposition is a consequence of Theorem 2.20 actually. \square

3. LOCAL DECAY ESTIMATE AND JENSEN-KATO DECAY ESTIMATE FOR e^{-itH}

In this section, we will prove the Jensen-Kato decay estimate which depends on the asymptotic properties of $E'(\lambda)$ for λ small and large. Before doing this we give the following local decay estimate which is equivalent that $\langle x \rangle^{-\sigma}$ is H -smooth. H -smooth theory has many deep connections with scattering theory and spectral analysis, especially for Schrödinger operator with repulsive potential. As for the more backgrounds of H -smooth theory, we refer the readers to see Reed and Simon [51, P. 344, XIII.7]. Notice that the local decay estimate shows the time-space integrability of the propagator e^{-itH} , which is also the key point of the Georgescu-Larenas-Soffer conjugate operator method.

Theorem 3.1. *Let $H = (-\Delta)^2 + V$ and $\langle x \rangle^\beta V(x) \in L^\infty(\mathbb{R}^d)$ with $\beta > 11 + \frac{3}{2}$ for $d = 3$, and $\beta > (d + 1)/2$ for $d \geq 5$. Assume that H has no positive embedded eigenvalues and 0 is a regular point. Then for any $\sigma > 1/2$ and $\phi \in L^2(\mathbb{R}^d)$, we have*

$$(3.1) \quad \int_{\mathbb{R}} \|\langle x \rangle^{-\sigma} e^{-itH} P_{ac} \phi\|_{L^2(\mathbb{R}^d)}^2 dt \leq C \|\phi\|_{L^2(\mathbb{R}^d)}^2,$$

where P_{ac} is the projection onto the absolutely continuous spectrum of H .

Proof. For $z \in \mathbb{C} \setminus [V_0, +\infty)$, by Proposition 2.15 we have as $z \rightarrow \infty$,

$$\|\langle x \rangle^{-\sigma} (H - z)^{-1} \langle x \rangle^{-\sigma}\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} = O(|z|^{-3/4}).$$

For $z \in \mathbb{C} \setminus [V_0, +\infty)$, by the low energy asymptotic of $R(H; z)$, we have as $z \rightarrow 0$

$$\|\langle x \rangle^{-\sigma} (H - z)^{-1} \langle x \rangle^{-\sigma}\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} = O(1), \quad d = 3;$$

$$\|\langle x \rangle^{-\sigma} (H - z)^{-1} \langle x \rangle^{-\sigma}\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} = O(1), \quad d \geq 5.$$

Then the conclusion follows by Corollary of [51, P.148]. \square

Now we begin to prove the Jensen-Kato decay estimate (see Theorem 1.1).

Proof of Theorem 1.1 (Jensen-Kato decay estimate). The specific values β, σ are based on the least needed expansion terms of the resolvent $R(H; z)$ around zero as shown in Proposition 2.8 and Theorem 2.7. Here we give the proof of 3-dimensional case. For $d \geq 5$ it follows by the similar argument.

Now, it suffice to prove that for any $u, \tilde{u} \in P_{ac} L^2(\mathbb{R}^3) \cap L_\sigma^2(\mathbb{R}^3)$,

$$|\langle \tilde{u}, e^{-itH} u \rangle| \leq C \langle t \rangle^{-5/4} \|\tilde{u}\|_{L_\sigma^2(\mathbb{R}^3)} \|u\|_{L_\sigma^2(\mathbb{R}^3)}.$$

For $u, \tilde{u} \in P_{ac} L^2(\mathbb{R}^3)$, by the spectral theorem we have

$$\langle \tilde{u}, e^{-itH} u \rangle = \int_0^\infty e^{-it\lambda} \langle \tilde{u}, E'(\lambda) u \rangle d\lambda = \int_0^\infty e^{-it\lambda} g(\lambda) d\lambda,$$

where $g(\lambda) = \langle \tilde{u}, E'(\lambda)u \rangle$. Notice that $g(\lambda)$ is smooth since $E^{(k+1)}(\lambda)$ is continuous in $\lambda \in (0, +\infty)$ for $k = 0, 1, 2, \dots$. Further, for any $\mathbb{N} \ni k \geq 0$, $g(\lambda)$ satisfies :

$$(3.2) \quad |g^{(k)}(\lambda)| = \left| \left\langle \langle x \rangle^\sigma \tilde{u}, \langle x \rangle^{-\sigma} E^{(k+1)}(\lambda) \langle x \rangle^{-\sigma} \langle x \rangle^\sigma u \right\rangle \right| \leq \|E^{(k+1)}(\lambda)\|_{\mathcal{B}(L_\sigma^2, L_{-\sigma}^2)} \|\tilde{u}\|_{L_\sigma^2} \|u\|_{L_\sigma^2}.$$

Let $\chi_l(\lambda)$ be a smooth cutoff function, i.e.

$$\chi_l \in C_0^\infty(\mathbb{R}), \quad \chi_l(\lambda) = \begin{cases} 1, & |\lambda| \leq \frac{1}{2}; \\ 0, & |\lambda| \geq 1. \end{cases}$$

Then $g(\lambda) = \chi_l(\lambda)g(\lambda) + (1 - \chi_l(\lambda))g(\lambda) = g_l(\lambda) + g_h(\lambda)$. Thus

$$(3.3) \quad \langle \tilde{u}, e^{-itH}u \rangle = \int_0^\infty e^{-it\lambda} g_l(\lambda) d\lambda + \int_0^\infty e^{-it\lambda} g_h(\lambda) d\lambda.$$

For the second integral of (3.3), it is the Fourier transform of $g_h(\lambda)$ clearly. Note that Proposition 2.22 and estimate (3.2) imply that for any positive integer k , $g_h^{(k)}(\lambda) \in L^1((0, \infty))$. Then the Riemann-Lebesgue's lemma tells that: for large t we have,

$$\begin{aligned} & \left| \int_0^\infty e^{-it\lambda} g_h(\lambda) d\lambda \right| \leq |t|^{-k} \|g_h^{(k)}(\lambda)\|_{L^1} \\ & \leq |t|^{-k} \int_{1/2}^\infty \|E^{(k+1)}(\lambda)\|_{\mathcal{B}(L_\sigma^2, L_{-\sigma}^2)} d\lambda \|\tilde{u}\|_{L_\sigma^2} \|u\|_{L_\sigma^2} \\ & \leq C|t|^{-k} \|\tilde{u}\|_{L_\sigma^2} \|u\|_{L_\sigma^2}. \end{aligned}$$

For the first integral of (3.3), integration by parts, we obtain

$$\int_0^\infty e^{-it\lambda} g_l(\lambda) d\lambda = \int_0^\infty \frac{e^{-it\lambda}}{it} g'_l(\lambda) d\lambda.$$

So that it remains to prove that

$$\int_0^\infty e^{-it\lambda} g'_l(\lambda) d\lambda = O(t^{-1/4}), \quad t \rightarrow \infty.$$

In fact, we have for large t

$$\begin{aligned} & \left| \int_0^\infty e^{-it\lambda} g'_l(\lambda) d\lambda \right| = \frac{1}{2} \left| \int_0^\infty e^{-it\lambda} (g'_l(\lambda + \pi/t) - g'_l(\lambda)) d\lambda \right| \\ & \leq \int_0^{\pi/t} |(g'_l(\lambda + \pi/t) - g'_l(\lambda))| d\lambda + \int_{\pi/t}^\infty |(g'_l(\lambda + \pi/t) - g'_l(\lambda))| d\lambda \\ & \leq 2 \int_0^{\pi/t} |g'_l(\lambda)| d\lambda + \int_{\pi/t}^\infty d\lambda \int_\lambda^{\lambda+\pi/t} |g'_l(\tilde{\lambda})| d\tilde{\lambda} = O(|t|^{-1/4}) \end{aligned}$$

by the asymptotic properties of $E'(\lambda)$ as in Proposition 2.21. \square

Corollary 3.2. *Let λ_j be the negative eigenvalues of H and P_j be the associated eigen-projection. Then under the same spectral assumption as in Theorem 1.1, we have in $\mathcal{B}(L_\sigma^2(\mathbb{R}^d), L_{-\sigma}^2(\mathbb{R}^d))$ with σ sufficient large:*

$$e^{-itH} - \sum_{j=1}^\# e^{-it\lambda_j} P_j - P_0 = t^{-5/4} C_1 + t^{-3/2} C_2 + \dots, \quad d = 3.$$

$$e^{-itH} - \sum_{j=1}^{\#} e^{-it\lambda_j} - P_0 = t^{-d/4} B + \dots, \quad d \geq 5.$$

Here $\# = \#\{\text{negative eigenvalues of } H\}$. And C_1, C_2, B can be calculate precisely from the expansion of $R(H_0; z)$ and $(1 + R(H_0; z)V)^{-1}$.

Remark 3.3. Although Murata had considered a general class of elliptic operator $P(D) + V$, but for $P(D) = (-\Delta)^m$ with $2 \leq m \in \mathbb{Z}^+$, his approach does not work. In fact, our method can apply to the higher order operator $(-\Delta)^m$. The resolvent $((-\Delta)^m - z)^{-1}$ can be expressed as

$$(3.4) \quad ((-\Delta)^m - z)^{-1} = \frac{1}{mz} \sum_{k=0}^{m-1} z_k (-\Delta - z_k)^{-1},$$

where $z_k = z^{\frac{1}{m}} e^{i\frac{2k\pi}{m}}$ ($k = 0, 1, 2, \dots, m-1$) are the k -th root of z , see [17]. According to this expression and under some suitable assumptions, one can obtain Jensen-Katon decay estimate of $(-\Delta)^m + V$ using the same strategy as the fourth-order operator $(-\Delta)^2 + V$.

4. L^p -TYPE DECAY ESTIMATES—GINIBRE ARGUMENT

4.1. L^p -boundedness of projection P_{ac} . In this section, we apply the iterated Duhamel formula to prove the $L^1 \rightarrow L^\infty$ in 3-dimension and $L^1 \cap L^2 \rightarrow L^2 + L^\infty$ in $d \geq 5$ for $e^{-itH} P_{ac}$. Let us first recall that Duhamel formula

$$(4.1) \quad e^{-itH} P_{ac} = e^{-itH_0} P_{ac} + i \int_0^t e^{-i(t-s)H_0} V P_{ac} e^{-isH} ds, \quad H = (-\Delta)^2 + V.$$

From the above formula, we need to give the L^p -boundedness of P_{ac} . For this end, let us summarize the spectrum of H . Firstly, we have assumed the absence of embedding positive eigenvalues of H , and zero is not an eigenvalue nor a resonance. So we have that $\sigma_c(H) = \sigma_{ac}(H) = [0, +\infty)$. Secondly, Birman and Solomyak's results [5] implies that there are only finite many discrete negative eigenvalues of H .

Denote the number of eigenvalues of H lying to the left of γ (counted according to their multiplicities) by $N(\gamma; H)$. The estimate of $N(\gamma; H)$ depends on the potential function. For the Schrödinger operator in \mathbb{R}^3 , Birman and Schwinger bound in [51] is the earliest estimate of $N(0; -\Delta + V)$ by

$$N(0; -\Delta + V) \leq \left(\frac{1}{4\pi}\right)^2 \int_{\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|} dx dy.$$

Later, Birman and Solomyak [5] discussed the operator $A_l(\alpha V) = (-\Delta)^l - \alpha V$, $l, \alpha > 0$. They had given the estimate of $N(\gamma; A_l(\alpha V))$ and discussed the asymptotic property of $N(\gamma; A_l(\alpha V))$ as α goes to infinite. For the fourth-order Schrödinger operator $H = (-\Delta)^2 + V$ in \mathbb{R}^d , It is known that

$$(4.2) \quad N(0; H) \leq C(d) \int_{\mathbb{R}^d} V_-^{d/4} dx, \quad d \geq 5.$$

For $d = 3$, according to Theorem 5.1 and Remarks in [5], we have for any γ positive,

$$(4.3) \quad N(-\gamma; H) \leq C \int_{\mathbb{R}^3} V_- dx, \quad d = 3.$$

Here V_- denotes the negative part of V . So that H has finitely many negative eigenvalues if the potential function decays fast enough.

On the decay of the eigenfunction, Deng, Ding and Yao [8] have established the pointwise kernel estimates for $e^{-t(P(D)+V)}$ with V belongs to the Kato potential class. One can prove that the eigenfunctions decay polynomially using the heat kernel estimate. Under some suitable assumptions on V and the fact that $P_{ac} = I - P_{disc}$, we have P_{ac} is $L^p \rightarrow L^p$ bounded for $1 \leq p \leq \infty$. Indeed, $P_{disc} = \sum_{j=0}^{\#} \langle \cdot, e_j \rangle e_j$ and the decay of eigenfunction e_j implies that P_{ac} is $L^p \rightarrow L^p$ bounded. Furthermore, we can prove that $\langle x \rangle^{-s} P_{ac} \langle x \rangle^s$ is also $L^p \rightarrow L^p$ bounded for any s positive by the same discussion.

Proposition 4.1. *Let $V(x) \in L^\infty(\mathbb{R}^d)$, and denote e_j to be the eigenfunction of H corresponding to eigenvalue λ_j . Then $e_j \in L_\sigma^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$ and any $\sigma \in \mathbb{R}^+$.*

Proof. We making use of the heat kernel $K(t, x, y)$ of e^{-tH} to prove the decay of eigenfunction. By the work of Deng, Ding and Yao [8, Theorem 1.1], we know that $K(t, x, y)$ satisfies

$$(4.4) \quad |K(t, x, y)| \leq C t^{-d/4} \exp \{ -c|x-y|^{4/3} t^{-1/3} + \theta t \}, \quad t > 0,$$

with some positive constants C, c, θ .

Since $e^{-H} e_j = e^{-\lambda_j} e_j$, then

$$\begin{aligned} \|\langle x \rangle^\sigma e_j\|_{L^p(\mathbb{R}^d)} &\lesssim \|\langle x \rangle^\sigma (e^{-|\cdot|^{4/3}} * e_j)\|_{L^p(\mathbb{R}^d)} = \left\| \int_{\mathbb{R}^d} \langle x \rangle^\sigma e^{-|x-y|^{4/3}} e_j(y) dy \right\|_{L^p(\mathbb{R}^d)} \\ &\lesssim \int_{\mathbb{R}^d} \|\langle x \rangle^\sigma e^{-|x-y|^{4/3}}\|_{L^p(\mathbb{R}^d)} |e_j(y)| dy = \int_{\mathbb{R}^d} \|\langle x \rangle^\sigma e^{-|x-y|^{4/3}} \langle y \rangle^\sigma\|_{L^p(\mathbb{R}^d)} |\langle y \rangle^{-\sigma} e_j(y)| dy \\ &\lesssim \int_{\mathbb{R}^d} \|\langle x \rangle^{2\sigma} e^{-|x-y|^{4/3}} (1 + |x-y|)^\sigma\|_{L^p(\mathbb{R}^d)} |\langle y \rangle^{-\sigma} e_j(y)| dy \\ &\lesssim \int_{\mathbb{R}^d} \|\langle x \rangle^{2\sigma} (1 + |x-y|)^{-\Pi}\|_{L^p(\mathbb{R}^d)} |\langle y \rangle^{-\sigma} e_j(y)| dy. \end{aligned}$$

Here we choose Π large enough such as $\Pi > 2\sigma + d + 1$. The last integral is finite by a simple discussion of the distance between x and y . \square

Theorem 4.2. *Let $V(x) \in L^\infty(\mathbb{R}^d)$ and such that integrals in (4.3) and (4.2) are convergent. For $\sigma \in \mathbb{R}$ and $1 \leq p \leq \infty$, under the same spectral assumption of Theorem 1.1, we have*

$$(4.5) \quad \|\langle x \rangle^{-\sigma} P_{ac} \langle x \rangle^\sigma f\|_{L^p(\mathbb{R}^d)} \leq c \|f\|_{L^p(\mathbb{R}^d)}, \quad f \in L^p(\mathbb{R}^d).$$

Proof. Since $P_{ac} = I - P_{disc}$, if the above inequality holds for P_{disc} then (4.2) holds. In fact $P_{disc} = \sum_{j=0}^{\#} \langle \cdot, e_j \rangle e_j$, and then

$$\begin{aligned} \|\langle \cdot \rangle^{-\sigma} P_{ac} \langle \cdot \rangle^\sigma f\|_{L^p(\mathbb{R}^d)} &= \left\| \sum_{j=0}^{\#} \langle \langle x \rangle^\sigma f, e_j \rangle \langle y \rangle^{-\sigma} e_j \right\|_{L^p(\mathbb{R}^d)} \\ &\lesssim \sum_{j=0}^{\#} |\langle f, \langle x \rangle^\sigma e_j \rangle| \|\langle y \rangle^{-\sigma} e_j\|_{L^p(\mathbb{R}^d)} \lesssim \sum_{j=0}^{\#} \|f\|_{L^p(\mathbb{R}^d)} \|\langle x \rangle^\sigma e_j\|_{L^{p'}(\mathbb{R}^d)} \|\langle y \rangle^{-\sigma} e_j\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

Thus Proposition 4.1 implies that the last sum is finite. \square

4.2. $L^1 \rightarrow L^\infty$ **decay estimate for $d = 3$** . Now, we give the proof of Theorem 1.2. Our strategy is applying the iterated Duhamel formula

$$(4.6) \quad \begin{aligned} e^{-itH} P_{ac} &= e^{-itH_0} P_{ac} + i \int_0^t e^{-i(t-s)H_0} V P_{ac} e^{-isH_0} ds \\ &\quad - \int_0^t \int_0^s e^{-i(t-s)H_0} V e^{-i(s-\tau)H} P_{ac} V e^{-i\tau H_0} d\tau ds \\ &:= I + II + III, \end{aligned}$$

and then estimate each term of (4.6). Notice that (1.2) and the L^p -boundedness of P_{ac} implies the $L^1 \rightarrow L^\infty$ estimate for the free term $e^{-itH_0} P_{ac}$.

$$(4.7) \quad \|I\| := \|e^{-itH_0} P_{ac} f\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/4} \|P_{ac} f\|_{L^1(\mathbb{R}^d)} \lesssim |t|^{-d/4} \|f\|_{L^1(\mathbb{R}^d)}.$$

For the second term II of (4.6), we have

$$\begin{aligned} \|II\| &:= \left\| \int_0^t e^{-i(t-s)H_0} V P_{ac} e^{-isH_0} u ds \right\|_{L^\infty(\mathbb{R}^3)} \\ &\lesssim \int_0^t (t-s)^{-3/4} \|V P_{ac} e^{-isH_0} u\|_{L^1(\mathbb{R}^3)} ds \\ &\lesssim \int_0^t (t-s)^{-3/4} \|V\|_{L^1(\mathbb{R}^3)} \|P_{ac}\|_{L^\infty \rightarrow L^\infty(\mathbb{R}^3)} \|e^{-isH_0} u\|_{L^\infty(\mathbb{R}^3)} ds \\ &\lesssim \int_0^t (t-s)^{-3/4} s^{-3/4} ds \|u\|_{L^1(\mathbb{R}^3)} \lesssim |t|^{-1/2} \|u\|_{L^1(\mathbb{R}^3)}. \end{aligned}$$

For the third term III of (4.6), we have

$$\begin{aligned} \|III\| &:= \left\| \int_0^t \int_0^s e^{-i(t-s)H_0} V e^{-i(s-\tau)H} P_{ac} V e^{-i\tau H_0} u d\tau ds \right\|_{L^\infty(\mathbb{R}^3)} \\ &\lesssim \int_0^t \int_0^s (t-s)^{-3/4} \|V \langle x \rangle^\sigma \langle x \rangle^{-\sigma} e^{-i(s-\tau)H} P_{ac} \langle x \rangle^{-\sigma} \langle x \rangle^\sigma V e^{-i\tau H_0} u\|_{L^1(\mathbb{R}^3)} d\tau ds \\ &\lesssim \int_0^t \int_0^s (t-s)^{-3/4} \|V \langle x \rangle^\sigma\|_{L^2(\mathbb{R}^3)} \|\langle x \rangle^{-\sigma} e^{-i(s-\tau)H} P_{ac} \langle x \rangle^{-\sigma} \langle x \rangle^\sigma V e^{-i\tau H_0} u\|_{L^2(\mathbb{R}^3)} d\tau ds \\ &\lesssim \int_0^t \int_0^s (t-s)^{-3/4} \|\langle x \rangle^{-\sigma} e^{-i(s-\tau)H} P_{ac} \langle x \rangle^{-\sigma}\|_{L^2 \rightarrow L^2(\mathbb{R}^3)} \|e^{-i\tau H_0} u\|_{L^\infty(\mathbb{R}^3)} d\tau ds \\ &\lesssim \int_0^t \int_0^s (t-s)^{-3/4} \langle s-\tau \rangle^{-5/4} \tau^{-3/4} d\tau ds \|u\|_{L^1(\mathbb{R}^3)} \lesssim \langle t \rangle^{-1/2} \|u\|_{L^1(\mathbb{R}^3)}. \end{aligned}$$

Therefore, we can combine the steps above to conclude the proof of Theorem 1.2.

4.3. $L^1 \cap L^2 \rightarrow L^2 + L^\infty$ **decay estimate for $d \geq 5$** . It is hard to obtain the $L^1 \rightarrow L^\infty$ estimate of the propagator e^{-itH} for operator H with potential in the dimension $d \geq 5$. Note that the $L^1 \rightarrow L^\infty$ -estimate time decay rate for $e^{-it\Delta^2}$ is $-d/4$, and the decay rate of Jensen-Kato estimate also is $-d/4$. For $e^{-itH} P_{ac}$ with $d \geq 5$, if we follow the same argument as $d = 3$, then the last two integrals of (4.6) are not convergent. Hence we will establish another type L^p decay estimate, i.e. the $L^1 \cap L^2 \rightarrow L^2 + L^\infty$ estimate, which is weaker than the $L^1 \rightarrow L^\infty$ estimate. Under such norm $\|\cdot\|_{L^1 \cap L^2 \rightarrow L^2 + L^\infty}$, it can cancel the singularity at zero of the two integrals of (4.6).

Definition 4.3. For any measurable function f , if $f = f_1 + f_2$ with $f_1 \in L^2(\mathbb{R}^d)$, $f_2 \in L^\infty(\mathbb{R}^d)$ and satisfies

$$\inf \left\{ \|f_1\|_{L^2(\mathbb{R}^d)} + \|f_2\|_{L^\infty(\mathbb{R}^d)} \right\} < \infty.$$

Here the infimum takes from all the splitting of f . Then we denote $f \in L^2 + L^\infty(\mathbb{R}^d)$, and $L^2 + L^\infty(\mathbb{R}^d)$ is a Banach space with the norm

$$\|f\|_{L^2 + L^\infty(\mathbb{R}^d)} = \inf \left\{ \|f_1\|_{L^2(\mathbb{R}^d)} + \|f_2\|_{L^\infty(\mathbb{R}^d)} \right\}.$$

Note that for $f \in L^2 \cap L^\infty(\mathbb{R}^d)$, since f can be divided as $f = f + 0 = 0 + f$, then $\|f\|_{L^2 + L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)}$ and $\|f\|_{L^2 + L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^\infty(\mathbb{R}^d)}$.

We start from the following lemma since we will face such kind of integral in the proof.

Lemma 4.4. For any $a > 0$ and $b > 0$, we have

$$(4.8) \quad \int_0^t \frac{ds}{\langle t-s \rangle^a \langle s \rangle^b} \leq \begin{cases} C \langle t \rangle^{-a-b+1}, & 0 < a, b < 1, \\ C \langle t \rangle^{-\min\{a, b\}}, & \text{otherwise.} \end{cases}$$

Proof. Since $c_1(1 + |x|) \leq \langle x \rangle \leq c_2(1 + |x|)$, c_1 and c_2 are some positive constants, then

$$\begin{aligned} & \int_0^t \frac{ds}{\langle t-s \rangle^a \langle s \rangle^b} \simeq \int_0^t \frac{ds}{(1 + (t-s))^a (1+s)^b} \\ &= \int_0^{t/2} \frac{ds}{(1 + (t-s))^a (1+s)^b} + \int_{t/2}^t \frac{ds}{(1 + (t-s))^a (1+s)^b} \\ &= \int_0^{t/2} \frac{ds}{(1 + (t-s))^a (1+s)^b} + \int_0^{t/2} \frac{d\tau}{(1 + \tau)^a (1 + (t-\tau))^b} \\ &\leq (1 + \frac{t}{2})^{-a} \int_0^{t/2} \frac{ds}{(1+s)^b} + (1 + \frac{t}{2})^{-b} \int_0^{t/2} \frac{d\tau}{(1+\tau)^a} \\ &= \frac{1}{1-b} (1 + \frac{t}{2})^{-a} [(1 + \frac{t}{2})^{-b+1} - 1] + \frac{1}{1-a} (1 + \frac{t}{2})^{-b} [(1 + \frac{t}{2})^{-a+1} - 1] \end{aligned}$$

Now, the inequality (4.8) holds by a simple discussion of the relationship between a, b and 1. The constant C only depends on a and b . \square

Now, we give the proof of $L^2 \cap L^1(\mathbb{R}^d) \rightarrow L^2 + L^\infty(\mathbb{R}^d)$ decay estimate for $d \geq 5$.

Proof of Theorem 1.3. Similarly, applying the iterated Duhamel formula (4.6) again. By the definition of $L^2 + L^\infty$, we have

$$\|e^{-itH_0} P_{ac} u\|_{L^2 + L^\infty(\mathbb{R}^d)} \leq \min \left\{ \|e^{-itH_0} P_{ac} u\|_{L^2(\mathbb{R}^d)}, \|e^{-itH_0} P_{ac} u\|_{L^\infty(\mathbb{R}^d)} \right\}.$$

Then for $0 < |t| \leq 1$, we have $\|e^{-itH_0} P_{ac} u\|_{L^2 + L^\infty(\mathbb{R}^d)} \leq \|u\|_{L^2(\mathbb{R}^d)}$. And for $|t| > 1$, by the estimate (1.2), we have $\|e^{-itH_0} P_{ac} u\|_{L^2 + L^\infty(\mathbb{R}^d)} \leq |t|^{-d/4} \|u\|_{L^1(\mathbb{R}^d)}$. Thus for the first free term I we have

$$(4.9) \quad \|e^{-itH_0} P_{ac} u\|_{L^2 + L^\infty(\mathbb{R}^d)} \lesssim \langle t \rangle^{-d/4} \|u\|_{L^1 \cap L^2(\mathbb{R}^d)}.$$

For the second term II of (4.6), we have

$$\begin{aligned}
& \int_0^t \|e^{-i(t-s)H_0} V P_{ac} e^{-isH_0} u\|_{L^2+L^\infty(\mathbb{R}^d)} ds \\
& \lesssim \int_0^t \langle t-s \rangle^{-d/4} \|V P_{ac} e^{-isH_0} u\|_{L^1 \cap L^2(\mathbb{R}^d)} ds \\
& \lesssim \int_0^t \langle t-s \rangle^{-d/4} \|V \langle x \rangle^\sigma\|_{L^\infty \cap L^2(\mathbb{R}^d)} \|\langle x \rangle^{-\sigma} P_{ac} e^{-isH_0} u\|_{L^2(\mathbb{R}^d)} ds \\
& \lesssim \int_1^t \langle t-s \rangle^{-d/4} \|\langle x \rangle^{-\sigma}\|_{L^2} \|P_{ac} e^{-isH_0} u\|_{L^\infty(\mathbb{R}^d)} ds \\
& \quad + \int_0^1 \langle t-s \rangle^{-d/4} \|\langle x \rangle^{-\sigma}\|_{L^\infty} \|P_{ac} e^{-isH_0} u\|_{L^2(\mathbb{R}^d)} ds \\
& \lesssim \int_1^t \langle t-s \rangle^{-d/4} s^{-d/4} ds \|u\|_{L^1 \cap L^2(\mathbb{R}^d)} + \int_0^1 \langle t-s \rangle^{-d/4} 2^{d/4} \langle s \rangle^{-d/4} ds \|u\|_{L^1 \cap L^2(\mathbb{R}^d)} \\
& \lesssim \langle t \rangle^{-d/4} \|u\|_{L^2 \cap L^1(\mathbb{R}^d)}.
\end{aligned}$$

For the third term III of (4.6), we have

$$\begin{aligned}
& \int_0^t \int_0^s \|e^{-i(t-s)H_0} V e^{-i(s-\tau)H} P_{ac} V e^{-i\tau H_0} u\|_{L^2+L^\infty(\mathbb{R}^d)} d\tau ds \\
& \lesssim \int_0^t \int_0^s \langle t-s \rangle^{-d/4} \|V e^{-i(s-\tau)H} P_{ac} V e^{-i\tau H_0} u\|_{L^1 \cap L^2(\mathbb{R}^d)} d\tau ds \\
& \lesssim \int_0^t \int_0^s \langle t-s \rangle^{-d/4} \|V \langle x \rangle^\sigma\|_{L^\infty \cap L^2(\mathbb{R}^d)} \|\langle x \rangle^{-\sigma} e^{-i(s-\tau)H} P_{ac} V e^{-i\tau H_0} u\|_{L^2(\mathbb{R}^d)} d\tau ds \\
& \lesssim \int_0^t \int_0^s \langle t-s \rangle^{-d/4} \|\langle x \rangle^{-\sigma} e^{-i(s-\tau)H} P_{ac} \langle x \rangle^{-\sigma}\|_{L^2 \rightarrow L^2(\mathbb{R}^d)} \|\langle x \rangle^\sigma V e^{-i\tau H_0} u\|_{L^2(\mathbb{R}^d)} d\tau ds \\
& \lesssim \int_0^t \int_0^1 \langle t-s \rangle^{-d/4} \langle s-\tau \rangle^{-d/4} \|\langle x \rangle^\sigma V\|_{L^\infty(\mathbb{R}^d)} \|e^{-i\tau H_0} u\|_{L^2(\mathbb{R}^d)} d\tau ds \\
& \quad + \int_0^t \int_1^s \langle t-s \rangle^{-d/4} \langle s-\tau \rangle^{-d/4} \|\langle x \rangle^\sigma V\|_{L^2(\mathbb{R}^d)} \|e^{-i\tau H_0} u\|_{L^\infty(\mathbb{R}^d)} d\tau ds \\
& \lesssim \int_0^t \int_0^1 \langle t-s \rangle^{-d/4} \langle s-\tau \rangle^{-d/4} \|u\|_{L^2(\mathbb{R}^d)} d\tau ds \\
& \quad + \int_0^t \int_1^s \langle t-s \rangle^{-d/4} \langle s-\tau \rangle^{-d/4} \tau^{-d/4} \|u\|_{L^1(\mathbb{R}^d)} d\tau ds \\
& \lesssim \int_0^t \int_0^1 \langle t-s \rangle^{-d/4} \langle s-\tau \rangle^{-d/4} 2^{d/4} \langle \tau \rangle^{-d/4} \|u\|_{L^1 \cap L^2(\mathbb{R}^d)} d\tau ds \\
& \quad + \int_0^t \int_1^s \langle t-s \rangle^{-d/4} \langle s-\tau \rangle^{-d/4} \tau^{-d/4} \|u\|_{L^1 \cap L^2(\mathbb{R}^d)} d\tau ds \\
& \lesssim \langle t \rangle^{-d/4} \|u\|_{L^1 \cap L^2(\mathbb{R}^d)}.
\end{aligned}$$

Thus again, we can combine the steps above to conclude the proof of Theorem 1.3. \square

5. ENDPOINT STRICHARTZ ESTIMATES FOR $d \geq 5$

According to Keel-Tao's method [27], Strichartz estimate can be obtained from the $L^1 \rightarrow L^\infty$ decay estimate of e^{-itH} . For free operator H_0 , we can get the $L^1 \rightarrow L^\infty$ decay estimate by Fourier transform. But for the perturbed operator $H = H_0 + V$ it is much harder. Here, we obtain the $L^1 \rightarrow L^\infty$ estimate with t small and large separately in 3-dimensions, and then give the Strichartz estimate local in time. For $d \geq 5$, we apply the local decay estimate to derive the endpoint Strichartz estimate. Note that the endpoint pair are $q = 2$ and $r' = 2d/(d-4)$ for $d \geq 5$, the same argument of $d \geq 5$ does not work for 3-dimensional case. In this section, we first give the result of $d = 3$ and then give the proof of Theorem 1.4 for $d \geq 5$.

Proposition 5.1. *Let $d = 3$ and H satisfies the same conditions as in Theorem 1.2, and $1 \leq q < \infty$, $2 \leq r \leq \infty$ satisfies $4/q + 6/r > 3$. Then for finite positive number T , we have*

$$(5.1) \quad \|e^{-itH} P_{ac} u\|_{L_t^q L_x^r([0,T] \times \mathbb{R}^3)} \leq CT^{\mathfrak{I}} \|u\|_{L^{r'}(\mathbb{R}^3)},$$

where $\mathfrak{I} = \frac{1}{q} + \frac{3}{2r} - \frac{3}{4}$.

Proof. First, by the iterated Duhamel formula (4.6), scaling on time partially, we have

$$\begin{aligned} e^{-itH} P_{ac} &= e^{-i(t/T)(TH_0)} P_{ac} + i \int_0^t e^{-i((t-s)/T)(TH_0)} V P_{ac} e^{-i(s/T)(TH_0)} ds \\ &\quad - \int_0^t \int_0^s e^{-i(t-s)H_0} V e^{-i(s-\tau)H} P_{ac} V e^{-i\tau H_0} d\tau ds \\ &:= I' + II' + III. \end{aligned}$$

Following the same argument, we have

$$\|I'\| \lesssim |t/T|^{-3/4}, \quad \|II'\| \lesssim |t/T|^{-1/2}, \quad \|III\| \lesssim \langle t \rangle^{-1/2}.$$

Thus we have

$$\|e^{-itH} P_{ac}\|_{L^1(\mathbb{R}^3) \rightarrow L^\infty(\mathbb{R}^3)} \lesssim |t|^{-3/4}, \quad 0 < |t| < T.$$

Finally, we can obtain the estimate by taking interpolation of the $L^1 \rightarrow L^\infty$ estimate and $L^2 \rightarrow L^2$ estimate of $e^{-itH} P_{ac}$. \square

Now we will show how to apply local decay estimate to derive Strichartz estimate with $d \geq 5$. We start from the free case. For the free operator $H_0 = (-\Delta)^2$, by making use of the $L^1(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)$ decay estimate (1.2) and Keel-Tao's method [27], we have the following estimate.

Lemma 5.2. *For the free fourth-order Schrödinger equation in \mathbb{R}^d with $d \geq 5$, we have*

$$(5.2) \quad \|e^{-itH_0} u\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|u\|_{L^2(\mathbb{R}^d)},$$

$$(5.3) \quad \left\| \int_{s < t} e^{i(t-s)H_0} f(s) ds \right\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R} \times \mathbb{R}^d)}.$$

where $(q, r), (\tilde{q}', \tilde{r}')$ satisfy (1.12).

Proof of Theorem 1.4 (i.e. the global endpoint Strichartz estimates of e^{-itH}).

Proof. We divide the proof into the following several steps.

Step 1: We aim to show the homogeneous Strichartz estimate (1.13) and the dual homogeneous Strichartz estimate (1.14).

Let us consider the following equation

$$(5.4) \quad \begin{cases} i\partial_t \psi = H_0 \psi + V\psi, \\ \psi(0, \cdot) = \psi_0 \in L^2(\mathbb{R}^d). \end{cases}$$

For the homogeneous Strichartz estimate (1.13), using Duhamel formula, it is enough to show

$$\left\| \int_0^t e^{-i(t-s)H_0} V e^{-isH} P_{ac} \psi ds \right\|_{L_t^q L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|\psi\|_{L^2(\mathbb{R}^d)}.$$

In fact, by (5.3) and the local decay estimate (3.1), we have

$$\begin{aligned} & \left\| \int_0^t e^{-i(t-s)H_0} V e^{-isH} P_{ac} \psi ds \right\|_{L_t^q L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|V e^{-itH} P_{ac} \psi\|_{L_t^2 L_x^{\frac{2d}{d+4}}(\mathbb{R} \times \mathbb{R}^d)} \\ & \lesssim \|V \langle x \rangle^\sigma\|_{L^{d/2}(\mathbb{R}^d)} \|\langle x \rangle^{-\sigma} e^{-itH} P_{ac} \psi\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|\psi\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Further, the dual homogeneous Strichartz estimate (1.14) follows by the T^*T -method.

Step 2: We aim to show the retarded Strichartz estimate (1.15). The solution $\Psi(t, x)$ of equation (1.11) satisfies

$$P_{ac} \Psi(t, x) = e^{itH_0} P_{ac} \Psi_0 - i \int_0^t e^{i(t-s)H_0} V P_{ac} \Psi(s) ds - i \int_0^t e^{i(t-s)H_0} P_{ac} h(s) ds.$$

Then by Hölder inequality and step 1, we have

$$\begin{aligned} & \|P_{ac} \Psi(t, x)\|_{L_t^q L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)} \\ & \lesssim \|e^{itH_0} P_{ac} \Psi_0\|_{L_t^q L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)} + \left\| \int_0^t e^{i(t-s)H_0} P_{ac} h(s, \cdot) ds \right\|_{L_t^q L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)} \\ & \quad + \left\| \int_0^t e^{i(t-s)H_0} V P_{ac} \Psi(s) ds \right\|_{L_t^q L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)} \\ & \lesssim \|\Psi_0\|_{L^2} + \|h(t)\|_{L_t^{\frac{q'}{q}} L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)} + \|V P_{ac} \Psi(t)\|_{L_t^2 L_x^{\frac{2d}{d+4}}(\mathbb{R} \times \mathbb{R}^d)} \\ & \lesssim \|\Psi_0\|_{L^2} + \|h(t)\|_{L_t^{\frac{q'}{q}} L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)} + \|V \langle x \rangle^\sigma\|_{L^{d/2}(\mathbb{R}^d)} \|\langle x \rangle^{-\sigma} P_{ac} \Psi(t)\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^d)}. \end{aligned}$$

Now we show that

$$(5.5) \quad \|\langle x \rangle^{-\sigma} P_{ac} \Psi(t)\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|\Psi_0\|_{L^2(\mathbb{R}^d)} + \|h\|_{L_t^{\frac{q'}{q}} L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)}.$$

First, by Duhamel formula for Ψ , we have

$$\begin{aligned} & \|\langle x \rangle^{-\sigma} P_{ac} \Psi(t)\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|\langle x \rangle^{-\sigma} e^{itH} P_{ac} \Psi_0\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^d)} \\ & \quad + \left\| \int_0^t \langle x \rangle^{-\sigma} e^{i(t-s)H} P_{ac} h(s) ds \right\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^d)}. \end{aligned}$$

Then, we will finish the proof which only needs to show

$$(5.6) \quad \left\| \int_0^t \langle x \rangle^{-\sigma} e^{i(t-s)H} P_{ac} h(s) ds \right\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|\Psi_0\|_{L^2(\mathbb{R}^d)} + \|h\|_{L_t^{\frac{q'}{q}} L_x^{r'}(\mathbb{R} \times \mathbb{R}^d)}.$$

Step 3: We show the local decay estimate of the source term (5.6).

Consider the Cauchy problem

$$(5.7) \quad \begin{cases} i\partial_t \phi = H_0 \phi + h(t) = H\phi - V\phi + h(t), \\ \phi(0, \cdot) = \Psi_0. \end{cases}$$

Then Duhamel formula for the solution $\phi(t, x)$ reads

$$(5.8) \quad P_{ac}\phi(t) = e^{itH} P_{ac}\Psi_0 + i \int_0^t e^{i(t-s)H} P_{ac} V\phi(s) ds - i \int_0^t e^{i(t-s)H} P_{ac} h(s) ds.$$

For the left hand side of (5.8), since $\phi(t)$ is also a solution of $i\partial_t \phi = H_0 \phi + h(t)$, by (5.2), (5.3) and Duhamel formula again, we have

$$\begin{aligned} & \| \langle x \rangle^{-\sigma} P_{ac}\phi(t) \|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^d)} \\ & \lesssim \| \langle x \rangle^{-\sigma} P_{ac} e^{itH_0} \Psi_0 \|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^d)} + \left\| \langle x \rangle^{-\sigma} P_{ac} \int_0^t e^{i(t-s)H_0} h(s) ds \right\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^d)} \\ & \lesssim \| \langle x \rangle^{-\sigma} P_{ac} \langle x \rangle^\sigma \langle x \rangle^{-\sigma} e^{itH_0} \Psi_0 \|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^d)} \\ & \quad + \left\| \langle x \rangle^{-\sigma} P_{ac} \langle x \rangle^\sigma \langle x \rangle^{-\sigma} \int_0^t e^{i(t-s)H_0} h(s) ds \right\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^d)} \\ & \lesssim \| \Psi_0 \|_{L^2(\mathbb{R}^d)} + \left\| \int_0^t e^{i(t-s)H_0} h(s) ds \right\|_{L_t^2 L_x^{\frac{2d}{d-4}}(\mathbb{R} \times \mathbb{R}^d)} \\ & \lesssim \| \Psi_0 \|_{L^2(\mathbb{R}^d)} + \| h \|_{L_t^{\frac{d}{d-4}} L_x^{\frac{2d}{d-4}}(\mathbb{R} \times \mathbb{R}^d)}. \end{aligned}$$

Here, we apply Theorem 4.2, the boundedness of P_{ac} . The local decay estimate for the first term on the right hand side of (5.8) follows from (3.1).

For the second term of the right hand side of (5.8), notice that

$$\begin{aligned} & \left\| \int_0^t \langle x \rangle^{-\sigma} e^{i(t-s)H} P_{ac} V\phi(s) ds \right\|_{L_t^2 L_x^2(\mathbb{R} \times \mathbb{R}^d)} \\ & \lesssim \left\| \int_0^t \| \langle x \rangle^{-\sigma} e^{i(t-s)H} P_{ac} V\phi(s) \|_{L_x^2(\mathbb{R}^d)} ds \right\|_{L_t^2(\mathbb{R})} \\ & \lesssim \left\| \int_0^t \langle t-s \rangle^{-d/4} \| \langle x \rangle^\sigma V\phi(s) \|_{L_x^2(\mathbb{R}^d)} ds \right\|_{L_t^2(\mathbb{R})} \\ & \lesssim \| \Psi_0 \|_{L^2} + \| h \|_{L_t^{\frac{d}{d-4}} L_x^{\frac{2d}{d-4}}(\mathbb{R} \times \mathbb{R}^d)}. \end{aligned}$$

Thus the whole proof can be concluded. \square

6. JENSEN-KATO TYPE DECAY ESTIMATES—THE CONJUGATE OPERATOR METHOD

In this section we apply the abstract theory of decay estimates to the fourth order Schrödinger operator. The abstract theory of decay estimates was developed by Georgescu, Larenas and Soffer [36, 11]. This is a completely independent method of getting pointwise estimates in time depending on positive commutator techniques. For dispersive equations, linear or nonlinear, quantitative estimates of the decay rate of the solution is always needed. Here, we establish the pointwise decay estimate of Jensen-Kato type for H by this method. For the history of commutator method, we refer the readers to Amrein, Boutet de Monvel and Georgescu [2].

The conjugate operator $A := -\frac{i}{2}(x \cdot \nabla + \nabla \cdot x)$ and P_{ac} is the projection onto the space of absolutely continuous spectrum of H . By the abstract theory, we need to verify the following conditions:

- (a): H is of class $C^1(A)$;
 (b): $P_{ac}[H, iA]P_{ac} = P_{ac}(q(H) + K)P_{ac}$ for $K \equiv F^*E$ in the sense that

$$(\phi, P_{ac}[H, iA]P_{ac}\psi) = (P_{ac}\phi, q(H)P_{ac}\psi) + (FP_{ac}\phi, EP_{ac}\psi)$$
 for $\phi, \psi \in \mathcal{D}(H)$. $q(H)$ is a function of H , and E, F are Kato H -smooth on the range of P_{ac} ;
 (c): K is symmetric on $\mathcal{D}(H)$;
 (d): K is bounded on L^2 ;
 (e): $(\phi, P_{ac}[A, K]P_{ac}\psi) = (F'P_{ac}\phi, E'P_{ac}\psi)$, F', E' are Kato H -smooth on the range of P_{ac} .

Here $\mathcal{D}(H)$ be the form domain of H , and by the KLMN theorem we have

$$\mathcal{D}(H) = \mathcal{D}(H_0) = \mathcal{H}^1.$$

For (b), since

$$[H, iA] = 4H_0 - x \cdot \nabla V = 4H - (4V(x) + x \cdot \nabla V) = q(H) - K$$

So, $q(H) = 4H$ and $K_2 = 4V(x) + x \cdot \nabla V$. Further, $4V(x) + x \cdot \nabla V$ is a real valued function, so (c) holds. Rewrite $K_2 = \text{sgn}(K)|K|^{\frac{1}{2}}|K|^{\frac{1}{2}}$, then $(|K|^{\frac{1}{2}})^* = |K|^{\frac{1}{2}}$. Then take $E = \text{sgn}(K)|K|^{\frac{1}{2}}$ and $F = |K|^{\frac{1}{2}}$, so E and F are bounded operators on L^2 , and then (d) holds. For (e), since $[A, K] = -i(x \cdot \nabla V)(4 + x \cdot \nabla V)$, so E' and F' are bounded operators on L^2 by the assumptions on the potential V . From the assumptions on V , $x \cdot \nabla V$, Theorem 3.1 implies that E, F, E' and F' are Kato H -smooth. Now, we aim to show the first one.

Proposition 6.1. *For measurable function f and operator $\Lambda = -i\nabla_x$, then*

$$(6.1) \quad e^{-itA}f(\Lambda)e^{itA} = f(e^{-t}\Lambda).$$

Proof. Since $A = \frac{1}{2}(x \cdot \Lambda + \Lambda \cdot x)$, and for any $\varphi(x) \in L^2(\mathbb{R}^d)$, we have

$$(e^{-itA}\varphi)(x) = e^{-dt/2}\varphi(e^{-t}x),$$

and then

$$\begin{aligned} [e^{-itA}f(\Lambda)e^{itA}\varphi]^\wedge(\xi) &= e^{-itA}[f(\xi)e^{dt/2}\widehat{\varphi}(e^t\xi)] \\ &= e^{-dt/2}f(e^{-t}\xi)e^{dt/2}\widehat{\varphi}(e^{-t} \cdot e^t\xi) \\ &= f(e^{-t}\xi)\widehat{\varphi}(\xi), \end{aligned}$$

where $\widehat{\varphi}(\xi)$ is the Fourier transform of $\varphi(x)$. □

Proposition 6.2. H_0 is of class $C^1(A)$.

Proof. By the definition of $C^1(A)$, we need to show the operator valued function

$$G_0(t) = e^{itA}(H_0 - i)^{-1}e^{-itA} \in C^1(\mathcal{B}(L^2(\mathbb{R}^d)))$$

for $t \in \mathbb{R}$. By Proposition 6.1, we know

$$G_0(t) = (e^{-4t}\Delta^2 - i)^{-1}, G'_0(t) = (4e^{-4t}\Delta^2)(e^{-4t}\Delta^2 - i)^{-2}.$$

We need to check that

$$\begin{aligned} \|G'_0(t)\|_{L^2 \rightarrow L^2} &= \left\| \frac{4e^{-4t}|\xi|^4}{(e^{-4t}|\xi|^4 - i)^2} \right\|_{L^2 \rightarrow L^2} \\ &\leq 4 \left\| \frac{1}{e^{-4t}|\xi|^4 - i} \right\|_{L^2 \rightarrow L^2} + 4 \left\| \frac{1}{(e^{-4t}|\xi|^4 - i)^2} \right\|_{L^2 \rightarrow L^2} \leq C. \end{aligned}$$

Similarly, we have $\|G_0''(t)\|_{L^2 \rightarrow L^2} \leq C'$. Here C, C' are positive constants. \square

Proposition 6.3. *Suppose the potential function $V(x)$ satisfies $x \cdot \nabla V \in L^\infty(\mathbb{R}^d)$, then*

$$H = H_0 + V \in C^1(A).$$

Proof. By the definition of $C^1(A)$, we need to check the function

$$G(t) = e^{itA}(H - i)^{-1}e^{-itA} \in C^1(\mathcal{B}(L^2(\mathbb{R}^d)))$$

for $t \in \mathbb{R}$. By the second resolvent formula, we have

$$\begin{aligned} G(t) &= e^{itA}[(H_0 - i)^{-1} - (H - i)^{-1}V(H_0 - i)^{-1}]e^{-itA} \\ &= e^{itA}(H_0 - i)^{-1}e^{-itA} - e^{itA}(H - i)^{-1}e^{-itA}e^{itA}Ve^{-itA}e^{itA}(H_0 - i)^{-1}e^{-itA} \\ &= G_0(t) - G(t)\tilde{V}(t)G_0(t), \end{aligned}$$

Here $\tilde{V}(t)$ denotes $e^{itA}Ve^{-itA}$. So we have $G(t)[1 + G_0(t)\tilde{V}(t)] = G_0(t)$. While by Proposition 2.14, the operator

$$1 + G_0(t)\tilde{V}(t) = e^{itA}[1 + (H_0 - i)^{-1}V]e^{-itA}$$

is invertible, thus we get the relationship between $G(t)$ and $G_0(t)$,

$$(6.2) \quad G(t) = G_0(t)[1 + G_0(t)\tilde{V}(t)]^{-1}.$$

Further,

$$(6.3) \quad G'(t) = G_0'(t)[1 + G_0(t)\tilde{V}(t)]^{-1} - G_0(t)[1 + G_0(t)\tilde{V}(t)]^{-2}[G_0'(t)\tilde{V}(t) + G_0(t)\tilde{V}'(t)].$$

Since $H_0 \in C^1(A)$ equals $G_0(t), G_0'(t)$ are continuous, so if $\tilde{V}(t)$ and $\tilde{V}'(t)$ are continuous in $\mathcal{B}(L^2(\mathbb{R}^d))$, then the proof done. For $\tilde{V}(t)$, $V(x) \in L^\infty$ implies that $\tilde{V}(t)$ is continuous. For the second one, since

$$\tilde{V}'(t) = e^{itA}[iA, V]e^{-itA} = e^{itA}(x \cdot \nabla V)e^{-itA},$$

so by the assumption $\tilde{V}'(t)$ is continuous. \square

Denote \mathcal{E} be the collection

$$\mathcal{E} = \left\{ u \in \mathcal{D}(H) \mid \psi_u(t) := \langle u, e^{itH}u \rangle \in L_t^2(\mathbb{R}) \right\}.$$

It was shown that \mathcal{E} is a dense linear subspace of the absolute continuity subspace of H and $[u]_H = \|\psi_u\|_{L_t^2}^{\frac{1}{2}}$ is a complete norm on it, see [2, 12]. Note that

$$[u]_H^2 = \int_{\mathbb{R}} |\langle u, e^{itH}u \rangle|^2 dt = 2\pi \int_{\mathbb{R}} E'_u(\lambda)^2 d\lambda.$$

Through the theory of [36] and under our assumptions of V , one can construct a new conjugate operator $\tilde{A} = A + B$ (Larenas-Soffer conjugate operator), where B is the limit in $\mathcal{B}(L^2, L^2)$ as follows,

$$B = s - \lim_{t \rightarrow \infty} \int_0^t e^{-isH} P_{ac} K P_{ac} e^{isH} ds.$$

It's easy to check the bounded operator B exists and be well defined by local decay estimate (Theorem 3.1) with the potential $V(x)$ satisfies the same condition as in Theorem 3.1. Further, the Larenas-Soffer conjugate operator \tilde{A} satisfy $[\tilde{A}, H] = 4H$ and $H \in C^1(\tilde{A})$.

Theorem 6.4. *Under the same assumptions as given in Theorem 3.1, and V satisfies $x \cdot \nabla V \in L^\infty(\mathbb{R}^d)$. Then for $u \in \mathcal{D}(H) \cap \mathcal{D}(A)$ satisfying $\langle x \rangle^\sigma u \in L^2(\mathbb{R}^d)$ with $\sigma > 1/2$, we have*

$$(6.4) \quad |\psi_{P_{ac}u}(t)| = O(t^{-1/2}), \quad t \rightarrow \infty.$$

Proof. By [12, Corollary 8.2], we have

$$|\psi_{P_{ac}u}(t)| \leq c \langle t \rangle^{-\frac{1}{2}} \|\psi_{P_{ac}u}(t)\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|t\psi'_{P_{ac}u}(t)\|_{L^2(\mathbb{R})}^{\frac{1}{2}}.$$

Now, the aim is to check that $\psi_{P_{ac}u}(t)$ and $t\psi'_{P_{ac}u}(t)$ are in $L^2(\mathbb{R})$.

$$\begin{aligned} \|\psi_{P_{ac}u}(t)\|_{L^2(\mathbb{R})} &= \|\langle P_{ac}u, e^{itH} P_{ac}u \rangle\|_{L^2(\mathbb{R})} \\ &= \left\| \langle \langle x \rangle^\sigma P_{ac}u, \langle x \rangle^{-\sigma} e^{itH} P_{ac}u \rangle \right\|_{L^2(\mathbb{R})} \\ &\leq \left\| \|\langle x \rangle^\sigma P_{ac}u\|_{L_x^2} \|\langle x \rangle^{-\sigma} e^{itH} P_{ac}u\|_{L_x^2} \right\|_{L^2(\mathbb{R})} \\ &\leq c \|\langle x \rangle^\sigma P_{ac}u\|_{L_x^2} \|u\|_{L_x^2}. \end{aligned}$$

For $t\psi'_{P_{ac}u}(t)$, since $H \in C^1(A)$, through the theory of [36], one can construct a new conjugate operator \tilde{A} (Larenas-Soffer conjugate operator) such that $H \in C^1(\tilde{A})$ and $[H, i\tilde{A}] = 4H$. Further, the domain of \tilde{A} satisfies $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$. Since

$$\begin{aligned} 4it\psi'_{P_{ac}u}(t) &= 4i\langle P_{ac}u, itHe^{itH} P_{ac}u \rangle = \langle P_{ac}u, 4tHe^{itH} P_{ac}u \rangle \\ &= \langle P_{ac}u, [e^{itH}, \tilde{A}] \rangle \\ &= \langle P_{ac}u, e^{itH} \tilde{A} P_{ac}u \rangle - \langle e^{-itH} \tilde{A} P_{ac}u, P_{ac}u \rangle \\ &= \langle \langle x \rangle^{-\sigma} e^{itH} \tilde{A} P_{ac}u, \langle x \rangle^\sigma P_{ac}u \rangle - \langle \langle x \rangle^\sigma P_{ac}u, \langle x \rangle^{-\sigma} e^{itH} \tilde{A} P_{ac}u \rangle \end{aligned}$$

And then

$$\begin{aligned} \|t\psi'_{P_{ac}u}(t)\|_{L^2(\mathbb{R})} &\leq \frac{1}{2} \left\| \langle \langle x \rangle^{-\sigma} e^{-itH} \tilde{A} P_{ac}u, \langle x \rangle^\sigma P_{ac}u \rangle \right\|_{L^2(\mathbb{R})} \\ &\leq \frac{1}{2} \left\| \|\langle x \rangle^\sigma P_{ac}u\|_{L_x^2} \|\langle x \rangle^{-\sigma} e^{itH} \tilde{A} P_{ac}u\|_{L_x^2} \right\|_{L^2(\mathbb{R})} \\ &\leq \frac{c}{2} \|\tilde{A} P_{ac}u\|_{L_x^2} \|\langle x \rangle^\sigma u\|_{L_x^2}. \end{aligned}$$

Here we use the local decay estimates of H . Furthermore,

$$|\psi_{P_{ac}u}(t)| \leq c \langle t \rangle^{-\frac{1}{2}} \|\langle x \rangle^\sigma u\|_{L_x^2} \|Au\|_{L_x^2}^{\frac{1}{2}} \|u\|_{L_x^2}^{\frac{1}{2}}.$$

□

Remark 6.5. *For high energy of H , we can get faster decay for $\psi_{P_{ac}u}(t)$, and this coincide with the perturbation method. In fact, we have*

$$|t\langle P_{ac}u, \chi_{\geq 1}(H) e^{-itH} P_{ac}u \rangle| = |\chi_{\geq 1}(H) [H, \tilde{A}]^{-1} P_{ac}u, [e^{-itH}, \tilde{A}] P_{ac}u|.$$

It is easy to check

$$\chi_{\geq 1}(H) [H, \tilde{A}]^{-1} = \chi_{\geq 1}(H) (4H)^{-1} \in C^1(\tilde{A})$$

and bounded, so that operator $\chi_{\geq 1}(H) [H, \tilde{A}]^{-1}$ keeps $\mathcal{D}(\tilde{A})$ the domain of \tilde{A} .

Let $\tilde{u} = \chi_{\geq 1}(H) [H, \tilde{A}]^{-1} P_{ac}u$, and then we have

$$\begin{aligned} |\langle \tilde{u}, [e^{-itH}, \tilde{A}] P_{ac}u \rangle| &\lesssim |\langle \langle x \rangle^\sigma \tilde{u}, \langle x \rangle^{-\sigma} e^{-itH} \tilde{A} P_{ac}u \rangle| + |\langle \langle x \rangle^\sigma u, \langle x \rangle^{-\sigma} e^{itH} P_{ac} \tilde{A} \tilde{u} \rangle| \\ &\lesssim \|\tilde{A} P_{ac}u\|_{L^2(\mathbb{R}^d)} \|\langle x \rangle^\sigma u\|_{L^2(\mathbb{R}^d)} < \infty. \end{aligned}$$

We can apply the same argument to get much higher decay of high energy part by an iterated process.

From the expansion of the perturbed resolvent, and through the classical Jensen-Kato's work [22], we expect the time decay rate should be $-5/4$ in the 3-dimension. Next, we improve our results by iteration of the previous argument. For higher dimensions $d \geq 5$, we also can improve the decay rate through the same way as what we did in the 3-dimensional case.

Proposition 6.6. *Suppose $V(x)$ satisfies $x \cdot \nabla(x \cdot \nabla V) \in L^\infty(\mathbb{R}^d)$, then $H \in C^2(A)$.*

Proof. By the definition of $C^2(A)$, we need to prove

$$G(t) = e^{itA}(H - i)^{-1}e^{-itA} \in C^2(\mathcal{B}(L^2(\mathbb{R}^3))).$$

Since $H \in C^1(A)$, so we only need to check $G''(t)$ is continuous. By (6.3) we have

$$\begin{aligned} G''(t) &= G_0''(t)[1 + G_0(t)\tilde{V}(t)]^{-1} - 2G_0'(t)[1 + G_0(t)\tilde{V}(t)]^{-2}[G_0'(t)\tilde{V}(t) + G_0(t)\tilde{V}'(t)] \\ &\quad + 2G_0(t)[1 + G_0(t)\tilde{V}(t)]^{-3}[G_0'(t)\tilde{V}(t) + G_0(t)\tilde{V}'(t)]^2 \\ &\quad - G_0(t)[1 + G_0(t)\tilde{V}(t)]^{-2}[G_0''(t)\tilde{V}(t) + 2G_0'(t)\tilde{V}'(t) + G_0(t)\tilde{V}''(t)]. \end{aligned}$$

Thus, if $G_0''(t)$ and $\tilde{V}''(t)$ are continuous then the proof done. For $G_0''(t)$, it's automatically continuous since we have already proved that $H_0 \in C^k(A)$. For $\tilde{V}''(t)$, since

$$\tilde{V}''(t) = e^{itA}[iA, x \cdot \nabla V]e^{-itA} = e^{itA}[x \cdot \nabla(x \cdot \nabla V)]e^{-itA},$$

thus $x \cdot \nabla(x \cdot \nabla V) \in L^\infty$ implies that $\tilde{V}''(t) \in C(\mathcal{B}(L^2(\mathbb{R}^3)))$. \square

Remark 6.7. *The condition $H \in C^1(A)$ can be replaced by $x \cdot \nabla V \in L^\infty$. Further, since for $k \in \mathbb{N}^+$ we have $H_0 \in C^k(A)$, and then we can prove that $H \in C^k(A)$ by the same process with the assumption that $(x \cdot \nabla)^{k-1}(x \cdot \nabla V) \in L^\infty$. And under this condition and some suitable assumption of the vector u , one can get the time decay rate of the so-called pointwise decay estimates to any order by the commutator method [12, 36] and the following argument.*

Proposition 6.8. *Under the same assumptions as given in Theorem 3.1, and V satisfies $x \cdot \nabla(x \cdot \nabla V) \in L^\infty$. Then for $u \in \mathcal{D}(H)$ with the form of $u = |H|^{\frac{1}{2}}\tilde{u}$, $\tilde{u} \in \mathcal{D}(A^2) \cap \mathcal{D}(H)$, and $\langle x \rangle^\sigma \tilde{u}, \langle x \rangle^\sigma A\tilde{u} \in L^2(\mathbb{R}^3)$ with $\sigma > \frac{1}{2}$, we have*

$$(6.5) \quad |\psi_{P_{ac}u}| = O(t^{-3/2}), \quad t \rightarrow \infty.$$

Proof. By [12, Corollary 8.2], we have

$$|t\psi_{P_{ac}u}(t)| \leq c\langle t \rangle^{-\frac{1}{2}} \|t\psi_{P_{ac}u}(t)\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|t^2\psi'_{P_{ac}u}(t)\|_{L^2(\mathbb{R})}^{\frac{1}{2}}.$$

Now, our target is to prove that $t\psi_{P_{ac}u}(t)$ and $t^2\psi'_{P_{ac}u}(t)$ belong to $L_t^2(\mathbb{R})$. For this we use the LS conjugate operator \tilde{A} . For $t\psi_{P_{ac}u}(t)$, since

$$\begin{aligned} i4t\psi_{P_{ac}u}(t) &= i\langle P_{ac}u, 4te^{itH}P_{ac}u \rangle = i\langle P_{ac}|H|^{\frac{1}{2}}\tilde{u}, 4te^{itH}P_{ac}|H|^{\frac{1}{2}}\tilde{u} \rangle \\ &= i\langle P_{ac}\tilde{u}, 4tHe^{itH}\text{sgn}(H)P_{ac}\tilde{u} \rangle = \langle P_{ac}\tilde{u}, [e^{itH}, \tilde{A}]\text{sgn}(H)P_{ac}\tilde{u} \rangle \\ &= \langle P_{ac}\tilde{u}, e^{itH}\tilde{A}\text{sgn}(H)P_{ac}\tilde{u} \rangle - \langle e^{-itH}\tilde{A}P_{ac}\tilde{u}, \text{sgn}(H)P_{ac}\tilde{u} \rangle \end{aligned}$$

And then

$$\begin{aligned}
\|t\psi_{P_{ac}u}(t)\|_{L_t^2} &\leq \frac{1}{2} \left\| \langle e^{-itH} \tilde{A} P_{ac} \tilde{u}, \operatorname{sgn}(H) P_{ac} \tilde{u} \rangle \right\|_{L_t^2} \\
&= \frac{1}{2} \left\| \langle \langle x \rangle^{-\sigma} e^{-itH} \tilde{A} P_{ac} \tilde{u}, \langle x \rangle^{\sigma} \operatorname{sgn}(H) P_{ac} \tilde{u} \rangle \right\|_{L_t^2} \\
&\leq \frac{1}{2} \left\| \langle \langle x \rangle^{-\sigma} e^{-itH} \tilde{A} P_{ac} \tilde{u} \|_{L_x^2} \| \langle x \rangle^{\sigma} \operatorname{sgn}(H) P_{ac} \tilde{u} \|_{L_x^2} \right\|_{L_t^2} \\
&\leq c \frac{1}{2} \| \langle x \rangle^{\sigma} \tilde{u} \|_{L_x^2} \| \tilde{A} P_{ac} \tilde{u} \|_{L_x^2} \leq c \frac{1}{2} \| \langle x \rangle^{\sigma} \tilde{u} \|_{L_x^2} \| A \tilde{u} \|_{L_x^2}.
\end{aligned}$$

For $t^2 \psi'_{P_{ac}u}(t)$, since

$$\begin{aligned}
i16t^2 \psi'_{P_{ac}u}(t) &= \langle P_{ac} \tilde{u}, 4tH[e^{itH}, \tilde{A}] \operatorname{sgn}(H) P_{ac} \tilde{u} \rangle \\
&= \langle P_{ac} \tilde{u}, \tilde{\mathcal{A}}^2(e^{itH}) \operatorname{sgn}(H) P_{ac} \tilde{u} \rangle - \langle P_{ac} \tilde{u}, 4\tilde{\mathcal{A}}(e^{itH}) \operatorname{sgn}(H) P_{ac} \tilde{u} \rangle.
\end{aligned}$$

Here $\tilde{\mathcal{A}}(e^{itH}) = i[e^{itH}, \tilde{A}]$ and

$$\begin{aligned}
\langle P_{ac} \tilde{u}, \tilde{\mathcal{A}}^2(e^{itH}) P_{ac} \tilde{u} \rangle &= \langle e^{-itH} P_{ac} \tilde{u}, \tilde{A}^2 P_{ac} \tilde{u} \rangle \\
&\quad - 2\langle \tilde{A} P_{ac} \tilde{u}, e^{itH} \tilde{A} P_{ac} \tilde{u} \rangle + \langle \tilde{A}^2 P_{ac} \tilde{u}, e^{itH} P_{ac} \tilde{u} \rangle.
\end{aligned}$$

So the second term just the same as $4t\psi_{P_{ac}u}(t)$. Further, for the first term

$$\begin{aligned}
\| \langle P_{ac} \tilde{u}, \tilde{\mathcal{A}}^2(e^{itH}) P_{ac} \tilde{u} \rangle \|_{L_t^2} &\leq 2 \| \langle \tilde{A}^2 P_{ac} \tilde{u}, e^{itH} P_{ac} \tilde{u} \rangle \|_{L_t^2} + 2 \| \langle \tilde{A} P_{ac} \tilde{u}, e^{itH} \tilde{A} P_{ac} \tilde{u} \rangle \|_{L_t^2} \\
&\leq 2 \left\| \langle \langle x \rangle^{-\sigma} e^{-itH} \tilde{A}^2 P_{ac} \tilde{u}, \langle x \rangle^{\sigma} P_{ac} \tilde{u} \rangle \right\|_{L_t^2} \\
&\quad + 2 \left\| \langle \langle x \rangle^{-\sigma} e^{-itH} \tilde{A} P_{ac} \tilde{u}, \langle x \rangle^{\sigma} \tilde{A} P_{ac} \tilde{u} \rangle \right\|_{L_t^2} \\
&\leq 2 \left\| \langle \langle x \rangle^{-\sigma} e^{-itH} \tilde{A}^2 P_{ac} \tilde{u} \|_{L_x^2} \| \langle x \rangle^{\sigma} P_{ac} \tilde{u} \|_{L_x^2} \right\|_{L_t^2} \\
&\quad + 2 \left\| \langle \langle x \rangle^{-\sigma} e^{-itH} \tilde{A} P_{ac} \tilde{u} \|_{L_x^2} \| \langle x \rangle^{\sigma} \tilde{A} P_{ac} \tilde{u} \|_{L_x^2} \right\|_{L_t^2} \\
&\leq 2c \| \tilde{A}^2 P_{ac} \tilde{u} \|_{L_x^2} \| \langle x \rangle^{\sigma} P_{ac} \tilde{u} \|_{L_x^2} + 2c \| \tilde{A} P_{ac} \tilde{u} \|_{L_x^2} \| \langle x \rangle^{\sigma} \tilde{A} P_{ac} \tilde{u} \|_{L_x^2} \\
&\leq 2c \| A^2 \tilde{u} \|_{L_x^2} \| \langle x \rangle^{\sigma} \tilde{u} \|_{L_x^2} + 2c \| A \tilde{u} \|_{L_x^2} \| \langle x \rangle^{\sigma} A \tilde{u} \|_{L_x^2}.
\end{aligned}$$

Finally,

$$|\psi_{P_{ac}u}(t)| \leq c \langle t \rangle^{-\frac{3}{2}} \left\{ \| \langle x \rangle^{\sigma} \tilde{u} \|_{L_x^2}^{\frac{1}{2}} (\| A^2 \tilde{u} \|_{L_x^2}^{\frac{1}{2}} + \| A \tilde{u} \|_{L_x^2}^{\frac{1}{2}}) + \| A \tilde{u} \|_{L_x^2}^{\frac{1}{2}} \| \langle x \rangle^{\sigma} A \tilde{u} \|_{L_x^2}^{\frac{1}{2}} \right\}.$$

□

Theorem 6.9. *Under the same conditions of Proposition 6.8, let $u = |H|^{\frac{3}{8}} \tilde{u}$, then*

$$(6.6) \quad |\psi_{P_{ac}u}| = O(t^{-5/4}), \quad t \rightarrow \infty.$$

Proof. For $0 \leq \operatorname{Re}(z) \leq 1$ and $z \in \mathbb{C}$, define function

$$\phi(z) = \left| \langle |H|^{z/2} P_{ac} u, e^{itH} |H|^{z/2} P_{ac} u \rangle \right|.$$

Here we use that $HP_{ac} \geq 0$. While through (6.4) and (6.5) we know

$$\phi(0) \leq c_u \langle t \rangle^{-1/2}, \quad \phi(1) \leq C_u \langle t \rangle^{-3/2},$$

and then by Hadamard's three line lemma, we have

$$|\psi_{P_{ac}u}(t)| = \phi\left(\frac{3}{4}\right) \leq C'_u(\langle t \rangle^{-\frac{1}{2}})^{1-\frac{3}{4}}(\langle t \rangle^{-\frac{3}{2}})^{\frac{3}{4}} = C'_u\langle t \rangle^{-\frac{5}{4}}.$$

□

The abstract theory also can deal with functions of H . We will apply this theory to the operator $\sqrt{H+m^2}$ ($m > 0$ large enough such that $H+m^2 \geq 0$) to get the Jensen-Kato type decay estimates. The difficulty is to prove $\sqrt{H+m^2} \in C^1(\tilde{A})$.

Lemma 6.10. $\sqrt{H+m^2} \in C^1(\tilde{A})$.

Proof. By the definition of $C^1(\tilde{A})$, we need to show the operator valued function

$$g(t) = e^{it\tilde{A}}(\sqrt{H+m^2} - i)^{-1}e^{-it\tilde{A}} \in C^1(\mathcal{B}(L^2(\mathbb{R}^d))), \quad t \in \mathbb{R}.$$

First, since \tilde{A} is self-adjoint then

$$\|g(t)\|_{L^2 \rightarrow L^2} = \|(\sqrt{H+m^2} - i)^{-1}\|_{L^2 \rightarrow L^2} \leq 1.$$

Further, since we have proven that $H \in C^1(\tilde{A})$ and,

$$\begin{aligned} g'(t) &= -e^{it\tilde{A}}[(\sqrt{H+m^2} - i)^{-1}, i\tilde{A}]e^{-it\tilde{A}} \\ &= e^{it\tilde{A}}(\sqrt{H+m^2} - i)^{-1}[\sqrt{H+m^2}, i\tilde{A}](\sqrt{H+m^2} - i)^{-1}e^{-it\tilde{A}}. \end{aligned}$$

We claim that

$$e^{it\tilde{A}}[\sqrt{H+m^2}, i\tilde{A}]e^{-it\tilde{A}} \in C(\mathcal{B}(L^2(\mathbb{R}^d))).$$

In fact, by the square root formula [50, P. 316]

$$\sqrt{H+m^2} = \frac{1}{\pi} \int_0^\infty \frac{\lambda^{-1/2}}{\lambda + H + m^2} (H + 1) d\lambda,$$

and then

$$\begin{aligned} [\sqrt{H+m^2}, i\tilde{A}] &= \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} \left[\frac{1}{\lambda + H + m^2} (H + m^2), i\tilde{A} \right] d\lambda \\ &= \frac{1}{\pi} \int_0^\infty -\lambda^{-1/2} \left[\frac{1}{\lambda + H + m^2}, i\tilde{A} \right] d\lambda \\ &= \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} \frac{1}{\lambda + H + m^2} [H, i\tilde{A}] \frac{1}{\lambda + H + m^2} d\lambda \end{aligned}$$

Since $H \in C^1(\tilde{A})$, thus $g'(t)$ is bounded. □

Following the same argument as in Remark 6.5 for $\sqrt{H+m^2}$, we have

Theorem 6.11. *Under the same assumptions as given in Theorem 3.1, and V satisfies $x \cdot \nabla V \in L^\infty(\mathbb{R}^d)$. Then for $u \in \mathcal{D}(\sqrt{H+m^2}) \cap \mathcal{D}(A)$ and $\langle x \rangle^\sigma u \in L^2(\mathbb{R}^d)$ with $\sigma > 1/2$, we have*

$$(6.7) \quad |\tilde{\psi}_{P_{ac}u}(t)| = O(t^{-1/2}), \quad t \rightarrow \infty,$$

with $\tilde{\psi}_u(t) = \langle u, e^{-it\sqrt{H+m^2}} u \rangle$.

Proof. Denote $\tilde{H} = \sqrt{H + m^2}$, and then we divided $|\tilde{\psi}_{P_{ac}u}(t)|$ into high energy part and low energy part:

$$|\tilde{\psi}_{P_{ac}u}(t)| = |\langle P_{ac}u, \chi_{\geq 1}(H)e^{-it\tilde{H}}P_{ac}u \rangle + \langle P_{ac}u, \chi_{< 1}(H)e^{-it\tilde{H}}P_{ac}u \rangle|.$$

For the high energy part $\langle P_{ac}u, \chi_{\geq 1}(H)e^{-it\tilde{H}}P_{ac}u \rangle$, we have

$$\begin{aligned} & \left| t \langle P_{ac}u, \chi_{\geq 1}(H)e^{-it\tilde{H}}P_{ac}u \rangle \right| = \left| \langle P_{ac}u, \frac{\chi_{\geq 1}(H)}{[\tilde{H}, \tilde{A}]} t[\tilde{H}, \tilde{A}]e^{-it\tilde{H}}P_{ac}u \rangle \right| \\ &= \left| \langle P_{ac}u, \frac{\chi_{\geq 1}(H)}{[\tilde{H}, \tilde{A}]} [e^{-it\tilde{H}}, \tilde{A}]P_{ac}u \rangle \right| = \left| \langle P_{ac}u, \frac{\chi_{\geq 1}(H)\tilde{H}}{4H} [e^{-it\tilde{H}}, \tilde{A}]P_{ac}u \rangle \right| \\ &\leq \left| \langle \frac{\chi_{\geq 1}(H)\tilde{H}}{4H} P_{ac}u, e^{-it\tilde{H}}\tilde{A}P_{ac}u \rangle \right| + \left| \langle \tilde{A}\frac{\chi_{\geq 1}(H)\tilde{H}}{4H} P_{ac}u, e^{-it\tilde{H}}P_{ac}u \rangle \right| \\ &\lesssim \|u\|_{L^2(\mathbb{R}^d)} (\|\tilde{A}u\|_{L^2(\mathbb{R}^d)} + \|\tilde{A}\frac{\chi_{\geq 1}(H)\tilde{H}}{4H} P_{ac}u\|_{L^2(\mathbb{R}^d)}) < \infty. \end{aligned}$$

Here we used the fact that $\frac{\chi_{\geq 1}(H)\tilde{H}}{4H}$ is bounded in $L^2(\mathbb{R}^d)$. So that for the high energy part we have

$$(6.8) \quad \left| \langle P_{ac}u, \chi_{\geq 1}(H)e^{-it\tilde{H}}P_{ac}u \rangle \right| = O(|t|^{-1}), \quad t \rightarrow \infty.$$

For the low energy part, we claim that

$$(6.9) \quad \left| \langle P_{ac}u, \chi_{< 1}(H)e^{-it\tilde{H}}P_{ac}u \rangle \right| = O(|t|^{-1/2}), \quad t \rightarrow \infty.$$

Note that $\left| \langle P_{ac}u, \chi_{< 1}(H)e^{-it\tilde{H}}P_{ac}u \rangle \right| = \left| \langle P_{ac}u, \chi_{< 1}(H)e^{-it(\tilde{H}-m)}P_{ac}u \rangle \right|$, and then we use the same approach as in Theorem 6.4 to show:

$$\left| \langle P_{ac}u, \chi_{< 1}(H)e^{-it(\tilde{H}-m)}P_{ac}u \rangle \right| = O(|t|^{-1/2}), \quad t \rightarrow \infty.$$

Now, we need only to show that

$$(6.10) \quad \left| \langle P_{ac}u, \chi_{< 1}(H)e^{-it(\tilde{H}-m)}P_{ac}u \rangle \right| \in L_t^2(\mathbb{R}),$$

$$(6.11) \quad \left| t \langle P_{ac}u, \chi_{< 1}(H)(\tilde{H} - m)e^{-it(\tilde{H}-m)}P_{ac}u \rangle \right| \in L_t^2(\mathbb{R}).$$

In fact, we have

$$\begin{aligned} & \sup_{0 \leq |z| < \infty} \left\| \langle x \rangle^{-\sigma} \chi_{< 1}(H)(\tilde{H} - m - z)^{-1} P_{ac} \langle x \rangle^{-\sigma} \right\|_{L^2 \rightarrow L^2} \\ &\leq \sup_{0 \leq |z| < 1} \left\| \langle x \rangle^{-\sigma} \chi_{< 1}(H)(\tilde{H} - m - z)^{-1} P_{ac} \langle x \rangle^{-\sigma} \right\|_{L^2 \rightarrow L^2} \\ &\quad + \sup_{1 \leq |z| < \infty} \left\| \langle x \rangle^{-\sigma} \chi_{< 1}(H)(\tilde{H} - m - z)^{-1} P_{ac} \langle x \rangle^{-\sigma} \right\|_{L^2 \rightarrow L^2} \\ &= \sup_{0 \leq |z| < 1} \left\| \langle x \rangle^{-\sigma} \chi_{< 1}(H)(\tilde{H} + m + z)(H + m^2 - (m + z)^2)^{-1} P_{ac} \langle x \rangle^{-\sigma} \right\|_{L^2 \rightarrow L^2} \\ &\quad + \sup_{1 \leq |z| < \infty} \left\| \langle x \rangle^{-\sigma} \chi_{< 1}(H)(\tilde{H} - m - z)^{-1} P_{ac} \langle x \rangle^{-\sigma} \right\|_{L^2 \rightarrow L^2} < \infty. \end{aligned}$$

So that by the Corollary of [51, P.146], we have the local decay for the low energy part:

$$(6.12) \quad \int_{\mathbb{R}} \left\| \langle x \rangle^{-\sigma} \chi_{< 1}(H)e^{-it(\tilde{H}-m)}P_{ac} \right\|_{L^2 \rightarrow L^2}^2 dt < \infty.$$

Thus (6.10) follows by (6.12).

Observe that $[\tilde{H}, \tilde{A}] = \tilde{H} - m^2 \tilde{H}^{-1}$ since we have proved $\tilde{H} \in C^1(\tilde{A})$. For (6.11), we have

$$\begin{aligned} & \left| t \langle P_{ac} u, \chi_{<1}(H)(\tilde{H} - m)e^{-it(\tilde{H}-m)} P_{ac} u \rangle \right| \\ &= \left| \langle P_{ac} u, \chi_{<1}(H)(\tilde{H} - m)[\tilde{H} - m, \tilde{A}]^{-1} t [\tilde{H} - m, \tilde{A}] e^{-it(\tilde{H}-m)} P_{ac} u \rangle \right| \\ &= \left| \langle P_{ac} u, \chi_{<1}(H)\tilde{H}(\tilde{H} + m)^{-1} [e^{-it(\tilde{H}-m)}, \tilde{A}] P_{ac} u \rangle \right|. \end{aligned}$$

Following the same argument as what we did for $-it\psi'_{P_{ac}u}(t)$ in Theorem 6.4, and the estimate (6.12) again yields that the (6.11) holds. \square

7. APPENDIX

A1: Proof of Theorem 2.7

Lemma 7.1. ([24, 23, Corollary 2.2]) *Let $F \subset \mathbb{C}$ have zero as an accumulation point. Let $T(z), z \in F$ be a family of bounded operators of the form*

$$T(z) = T_0 + zT_1(z)$$

with $T_1(z)$ uniformly bounded as $z \rightarrow 0$. Suppose 0 is an isolated point of the spectrum of T_0 , and let S be the corresponding Riesz projection. If $T_0 S = 0$, then for sufficient small $z \in F$ the operator $\tilde{T}(z) : S\mathcal{H} \rightarrow S\mathcal{H}$ defined by

$$(7.1) \quad \tilde{T}(z) = \frac{1}{z}(S - S(T(z) + S)^{-1}S) = \sum_{j=0}^{\infty} (-1)^j z^j S [T_1(z)(T_0 + S)^{-1}]^{j+1} S$$

is uniformly bounded as $z \rightarrow 0$. The operator $T(z)$ has a bounded inverse in \mathcal{H} if and only if $\tilde{T}(z)$ has a bounded inverse in $S\mathcal{H}$, and in this case

$$(7.2) \quad T(z)^{-1} = (T(z) + S)^{-1} + \frac{1}{z}(T(z) + S)^{-1} S \tilde{T}(z)^{-1} S (T(z) + S)^{-1}.$$

Lemma 7.2. *Let $\langle x \rangle^\kappa V \in L^2(\mathbb{R}^3)$ with some κ ($\kappa > 9$) sufficient large and let p be the largest integer satisfying $\kappa > 2p + 5$, then $M(\mu) - \frac{(1+i)\alpha}{8\pi} P\mu^{-1} - U$ is a uniformly bounded operator valued function in*

$$E = \{\mu \mid \operatorname{Re} \mu > 0, |\mu| \leq 1\}$$

and has the following asymptotic expansion for small $\mu \in E$

$$(7.3) \quad M(\mu) = \frac{(1+i)\alpha}{8\pi} P\mu^{-1} + \sum_{j=0}^{p-1} M_j \mu^j + \mu^p \mathfrak{R}_0(\mu, |x-y|)$$

where $P = \alpha^{-1} \langle v, \cdot \rangle$, $\alpha = \|v\|^2$ and $M_0 - U, M_j, j = 1, 2, 3, \dots, p-1$ are integral operators given by the kernels

$$(M_0 - U)(|x-y|) = -\frac{1}{8\pi} v(x)|x-y|v(y),$$

$$M_j(|x-y|) = \frac{(-1)^j(1-i^j)}{8\pi(j+2)!} v(x)|x-y|^{j+1}v(y),$$

$$\mathfrak{R}_0(\mu, |x-y|) = v(x) \frac{1}{8\pi(p+2)!} \frac{1}{\mu^2|x-y|} \int_0^\mu (e^{it|x-y|} - e^{-t|x-y|})^{(p+3)} (\mu-t)^{p+2} dt v(y),$$

and $\mathfrak{R}_0(\mu)$ is uniformly bounded in norm. Further, the operators $M_0 - U, M_j$ are compact.

Proof. By (2.16) and (2.11), we have

$$\begin{aligned}
 M(\mu) &= U + v \left[\sum_{j=0}^{p+1} \frac{i^j - (-1)^j}{8\pi j!} \mu^{j-2} |x-y|^{j-1} + \mu^p \Re(\mu, |x-y|) \right] v \\
 &= U + v \left[\frac{1+i}{8\pi} \mu^{-1} + \sum_{j=2}^{p+1} \frac{i^j - (-1)^j}{8\pi j!} \mu^{j-2} |x-y|^{j-1} \right] v + v \mu^p \Re(\mu, |x-y|) v \\
 &= U + v \left[\frac{1+i}{8\pi} \mu^{-1} + \sum_{j=0}^{p-1} \frac{(-1)^j (1-i^j)}{8\pi (j+2)!} \mu^j |x-y|^{j+1} \right] v + v \mu^p \Re(\mu, |x-y|) v
 \end{aligned}$$

thus we get (7.3). Using the Hilbert-Schmidt norm, it's trivial to check the operators $M_0 - U, M_j$ are compact. \square

Proof of Theorem 2.7. Writing

$$(7.4) \quad M(\mu) = \frac{(1+i)\alpha}{8\pi\mu} (P + \mu \tilde{M}(\mu))$$

where

$$\begin{aligned}
 \tilde{M}(\mu) &= \frac{8\pi}{(1+i)\alpha} \left(\sum_{j=0}^{p-1} M_j \mu^j + \mu^p \Re_0(\mu, |x-y|) \right) \\
 (7.5) \quad &= \frac{8\pi}{(1+i)\alpha} (M_0 + \mu M_1 + \mu^2 M_2(\mu)).
 \end{aligned}$$

Applying Lemma 7.1 to $P + \mu \tilde{M}(\mu)$, for sufficiently small μ we have

$$(7.6) \quad M(\mu)^{-1} = \frac{8\pi\mu}{(1+i)\alpha} \left[(1 + \mu \tilde{M}(\mu))^{-1} + \mu^{-1} (1 + \mu \tilde{M}(\mu))^{-1} Q m(\mu)^{-1} Q (1 + \mu \tilde{M}(\mu))^{-1} \right],$$

where $Q = 1 - P$ and

$$\begin{aligned}
 m(\mu) &= \sum_{j=0}^{\infty} \mu^j (-1)^j Q \left[\frac{8\pi}{(1+i)\alpha} (M_0 + \mu M_1 + \mu^2 M_2(\mu)) \right]^{j+1} Q \\
 (7.7) \quad &= \frac{8\pi}{(1+i)\alpha} Q M_0 Q - \frac{8\pi}{(1+i)\alpha} \mu Q \left(\frac{8\pi}{(1+i)\alpha} M_0^2 - M_1 \right) Q + \mu^2 m_2(\mu) \\
 &= \frac{8\pi}{(1+i)\alpha} [m_0 + \mu m_1(\mu)].
 \end{aligned}$$

Notice that $M_0 = U - \frac{1}{8\pi} v(x) |x-y| v(y)$ which is the symmetric form of integral kernel of $I + (-\Delta)^{-2} V$. So that $Q M_0 Q$ is invertible in $L^2(\mathbb{R}^3)$ under the assumption that zero is a regular point of H . Hence we get the asymptotic expansion by expanding each terms of (7.6) in powers of μ . \square

Finally, we remark that, the invertibility of $Q M_0 Q$ is related to the spectral properties of H at zero. If $Q M_0 Q$ is not invertible, denote $S : QL^2(\mathbb{R}^3) \rightarrow QL^2(\mathbb{R}^3)$ be the orthogonal projection on $\ker Q M_0 Q$, since $Q M_0 Q$ is self-adjoint, we have $\dim S < \infty$. Then $Q M_0 Q + S$ is invertible, and by applying lemma 7.1 to $m_0 + \mu m_1(\mu)$ we get the inverse of $m(\mu)^{-1}$. In order to get the expansion of $m(\mu)^{-1}$, by the inverse formula (7.2) we need to get the expansion of $\tilde{M}_1(\mu)$ which can be calculated from the inverse of $m(\mu)^{-1}$. This is an iterative process but it will stops in finite

steps. Since we assume that $V(x) = O(|x|^{-\beta})$ as $|x| \rightarrow \infty$, there exists a μ_0 such that for $\iota \in (0, \mu_0)$ we have $\iota^4 \in \rho(H)$. Since H is selfadjoint, so $\limsup_{\iota \downarrow 0} \|\iota^4(H - \iota^4)^{-1}\| \leq 1$ and then from (2.18) we have

$$(7.8) \quad \limsup_{\iota \downarrow 0} \|\iota^4 M(\iota)^{-1}\| \leq \infty.$$

Since each iteration adds to the singularity of $M(\mu)^{-1}$, after a few steps the leading term must be invertible and the process stops, due to (7.8). For Schrödinger operator $-\Delta + V$, Jensen and Nenciu have proved that the dimension of the null space of QM_0Q is one in 1-dimension. The difficulty to analyze the structure of QM_0Q for H is caused by the dimension $d \geq 3$. Even if we consider such problem for H in 1-dimension, it is also difficult since H is a higher order operator.

A2: Proof of Lemma 2.14

Proof. Part 1: compactness of $R(H_0; z)V$ and $VR(H_0; z)$. It suffices to prove the compactness of the operator $VR_0(z)$ since the adjoint operator $[R(H_0; z)V]^* = VR(H_0; \bar{z})$ is also compact. For any $\epsilon > 0$ we can split $V(x)$ as

$$V(x) = V_\epsilon(x) + r_\epsilon(x),$$

where $V_\epsilon(x) \in C_0^\infty(\mathbb{R}^d)$ and $\sup_{x \in \mathbb{R}^d} |r_\epsilon(x)| \leq \epsilon$. Therefore, $VR(H_0; z) = V_\epsilon R(H_0; z) + r_\epsilon R(H_0; z)$, where

$$\|VR(H_0; z) - V_\epsilon R(H_0; z)\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \rightarrow 0, \quad \epsilon \rightarrow 0.$$

Therefore, the operator $VR(H_0; z) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is compact if each operator $V_\epsilon R(H_0; z) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is compact for $z \in \mathbb{C} \setminus [0, +\infty)$, see [61, P. 278].

It remains to prove the compactness of $V_\epsilon R(H_0; z)$ in $L^2(\mathbb{R}^d)$. For $c > 0$ denote

$$B(c) := \{V_\epsilon R(H_0; z)\psi : \|\psi\|_{L^2(\mathbb{R}^d)} \leq c\}.$$

By the definition of compact operator, we need check that

For any $c > 0$, the set $B(c)$ is contained in a compact subset of $L^2(\mathbb{R}^d)$.

First, note that for $z \in \mathbb{C} \setminus [0, \infty)$ and $\|\psi\|_{L^2(\mathbb{R}^d)} \leq c$ we have

$$\|R(H_0; z)\psi\|_{\mathcal{H}^4(\mathbb{R}^d)} \leq c_1 < \infty,$$

since the operator $R(H_0; z) : L^2(\mathbb{R}^d) \rightarrow \mathcal{H}^4(\mathbb{R}^d)$ is bounded, therefore for $\|\psi\|_{L^2(\mathbb{R}^3)} \leq c$,

$$(7.9) \quad \|V_\epsilon R(H_0; z)(z)\psi\|_{\mathcal{H}^4(\mathbb{R}^d)} \leq c_2 < \infty,$$

since the operator of multiplication by V_ϵ is continuous in Sobolev space $\mathcal{H}^4(\mathbb{R}^d)$. Finally,

$$(7.10) \quad \text{supp} V_\epsilon R(H_0; z)\psi \subset \text{supp} V_\epsilon,$$

where $\text{supp} V_\epsilon$ is a bounded set since $V_\epsilon \in C_0^\infty(\mathbb{R}^d)$. Now (7.9) and (7.10) imply that $B(c)$ is contained in a compact subset of $L^2(\mathbb{R}^d)$ by Sobolev embedding Theory.

Part 2: Fredholm theorem. We need to prove that there exists an operator inverse $(1 + R(H_0; z)V)^{-1}$, which is continuous in $L^2(\mathbb{R}^d)$.

Step 1: We claim that the equation $(H - z)\psi = 0$ for $\psi \in L^2(\mathbb{R}^d)$ admits only trivial solution $\psi = 0$. First, $(H - z)\psi = 0$ implies $H\psi = z\psi \in L^2(\mathbb{R}^d)$, hence $\psi \in \mathcal{H}^4(\mathbb{R}^d)$. Therefore,

$$((H - z)\psi, \psi) = (H\psi, \psi) - z(\psi, \psi).$$

If $z \in \mathbb{C} \setminus \mathbb{R}$, then for $\psi \neq 0$,

$$\operatorname{Im}((H - z)\psi, \psi) = -\operatorname{Im}z(\psi, \psi) \neq 0.$$

Since the scalar product $(H\psi, \psi)$ is real. Hence $(H - z)\psi \neq 0$ for $\psi \neq 0$. It remains to consider $\operatorname{Re}z < V_0$ (recall that $V_0 = \min\{V(x), x \in \mathbb{R}^d\}$), then $\operatorname{Re}((H - z)\psi, \psi) \geq (V_0 - z)(\psi, \psi) \neq 0$ for $\psi \neq 0$. Hence $(H - z)\psi \neq 0$ for $\psi \neq 0$.

Step 2: We prove that the equation $[1 + R(H_0; z)V]\psi = 0$ for $\psi \in L^2(\mathbb{R}^d)$ admits only zero solution. Indeed, the identity $[1 + R(H_0; z)V]\psi = 0$ implies $(H - z)\psi = 0$ by the Born decomposition formula. Hence $\psi = 0$ as in step 1. Now the operator $[1 + R(H_0; z)V]$ is invertible in $L^2(\mathbb{R}^d)$ by Fredholm Theorem (see e.g. [61, 50]). \square

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